$$\langle \phi \rangle = \int d^3p\, \phi\, f^0(\vec{p}) \Big/ \int d^3p\, f^0(\vec{p})$$

$$= \int d^3v\, P(\vec{v})\, \phi \quad ; \quad P(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-m\vec{v}^2/2k_B T}$$

Here T is the avg temperature since $\vec{j}_\varepsilon \propto \vec{\nabla} T$.

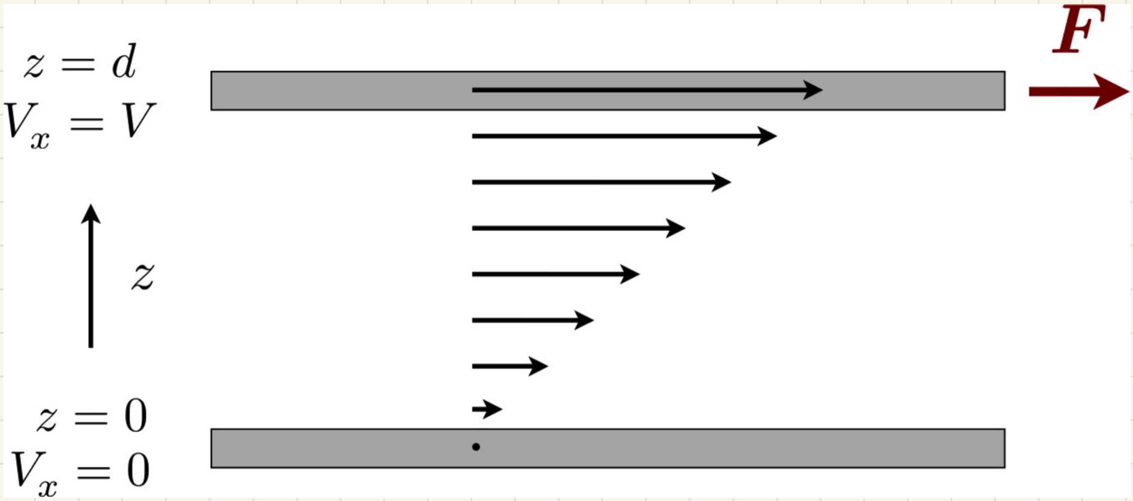
If $\phi = \phi(\varepsilon)$, we can use $P(\vec{v})\, d^3v = \mathcal{P}(\varepsilon)\, d\varepsilon$ with $\mathcal{P}(\varepsilon) = \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \varepsilon^{1/2} e^{-\varepsilon/k_B T}$. Then find $\langle \varepsilon^\theta \rangle = \frac{2}{\sqrt{\pi}} \Gamma(\theta + \frac{3}{2})(k_B T)^\theta$. Thus

$$\vec{j}_\varepsilon = -\kappa \vec{\nabla} T \quad , \text{ with thermal conductivity}$$

$$\kappa = \frac{2n\tau}{3mk_B T^2} \langle \varepsilon^2 (\varepsilon - \tfrac{5}{2} k_B T) \rangle = \frac{5n\tau k_B^2 T}{2m} = \frac{\pi}{8} n \ell \bar{v} c_p$$

Viscosity

Consider the situation depicted in the figure below. A fluid between two plates is sheared by applying a horizontal force to the upper plate. The lower plate is held fixed.



If the top plate moves at velocity $V\hat{x}$, what force $F\hat{x}$ must be applied? Fluid particles next to the top plate have average momentum $\langle p_x \rangle = mV$. As they move downward, they carry their $\hat{x}$-momentum with them, away from the top plate. Particles from below take their place and must be accelerated to $\langle \vec{v} \rangle = V\hat{x}$, resulting in a <u>viscous drag</u> on the upper plate, $\vec{F}_{drag}$, in the $-\hat{x}$ direction. Thus $\vec{F} = -\vec{F}_{drag}$. The momentum density injected into the fluid at the upper plate is then extracted at the lower plate. Avg momentum density is $\frac{1}{2}\rho V\hat{x}$.

<u>Momentum flux density</u> : $\Pi_{xz} = n\langle p_x v_z \rangle = \rho \langle v_x v_z \rangle$ is the drag force on the upper plate per unit area. Note $[\Pi_{xz}] = ML^{-3} \cdot (LT^{-1})^2 = ML^{-1}/T^2 = F/A$.

This is given by $\Pi_{xz} = -\eta \, \partial V_x / \partial_z$ where η is the <u>shear</u> <u>viscosity</u> of the fluid. For a fluid in motion with constant velocity $\vec{V}$, we have

$$\Pi_{\alpha\beta} = mn \langle (V_\alpha + v'_\alpha)(V_\beta + v'_\beta) \rangle$$
$$= nm V_\alpha V_\beta + \frac{1}{3} nm \langle \vec{v}'^2 \rangle$$
$$= \rho V_\alpha V_\beta + p \, \delta_{\alpha\beta}$$

where p is the pressure. When $\vec{V}$ is spatially varying,

$$\Pi_{\alpha\beta} = p \, \delta_{\alpha\beta} + \rho V_\alpha V_\beta - \tilde{\sigma}_{\alpha\beta}$$

where $\tilde{\sigma}_{\alpha\beta} = \eta \left(\frac{\partial V_\alpha}{\partial x^\beta} + \frac{\partial V_\beta}{\partial x^\alpha} - \frac{2}{3} \vec{\nabla} \cdot \vec{V} \delta_{\alpha\beta} \right) + \zeta \vec{\nabla} \cdot \vec{V} \delta_{\alpha\beta}$

is the <u>viscosity</u> <u>stress</u> <u>tensor</u>. Note that the term in green is $2\eta\left(Q_{\alpha\beta}-\frac{1}{3}\mathrm{Tr}\,Q\,\delta_{\alpha\beta}\right)$ where

$$Q_{\alpha\beta}=\frac{1}{2}\left(\frac{\partial V_\alpha}{\partial x_\beta}+\frac{\partial V_\beta}{\partial x_\alpha}\right) \quad \text{(as above)}$$

Thus $\mathrm{Tr}\,\tilde{\sigma}_{\alpha\beta}=3\zeta\,\vec{\nabla}\cdot\vec{V}$, where ζ is the <u>bulk viscosity</u>. Within the RTA,

$$\delta f(\vec{p})=-\frac{\tau}{k_B T}\left\{m v_\alpha v_\beta\,Q_{\alpha\beta}+\frac{\varepsilon-c_p T}{T}\,\vec{v}\cdot\vec{\nabla}T-\frac{k_B\varepsilon}{c_v}\,\vec{\nabla}\cdot\vec{V}\right\}f^0$$

Assuming $\vec{\nabla}T=\vec{\nabla}\cdot\vec{V}=0$, we have

$$\Pi_{xz}=n\int d^3p\;p_x v_z\,\delta f(\vec{p})=-\frac{n m^2\tau}{k_B T}\,Q_{\alpha\beta}\langle v_x v_z v_\alpha v_\beta\rangle$$

$$=-n\tau k_B T\left(\frac{\partial V_x}{\partial z}+\frac{\partial V_z}{\partial x}\right)$$

need $(\alpha,\beta)=(x,z)$ or (z,x) if this is to be nonzero

Thus we conclude

$$\eta=n k_B T\tau=\frac{\pi}{8}\,n m \ell\bar{v}\propto T^{1/2}$$

Note that $\eta(T)$ vanishes at $T=0$. Our intuition may say that a liquid should gum up at low T, as it approaches a glass or crystallization transition. But our calculation is for gases, and in a regime where such physics is not relevant.

Oscillating Force

Suppose the gas particles are subjected to an oscillating external force $\vec{F}(t) = \vec{F} e^{-i\omega t}$ (real part thereof). We'll keep things simple and take $\vec{\nabla} T = \partial_\alpha V_\beta = 0$, in which case the BE in the RTA is

$$\frac{\partial \delta f}{\partial t} + \vec{F} e^{-i\omega t} \cdot \underbrace{\vec{v} \frac{\partial f^0}{\partial \varepsilon}}_{\text{this is } \partial f^0 / \partial \vec{p}} = 0 \Rightarrow$$

$$\delta f(\vec{p}, t) = \frac{\tau e^{-i\omega t}}{1 - i\omega\tau} \frac{\partial f^0}{\partial \varepsilon} \vec{F} \cdot \vec{v}$$

Now compute the particle current:

$$J^\alpha(t) = \int d^3p \, v^\alpha \delta f = \frac{\tau e^{-i\omega t}}{1 - i\omega\tau} \frac{F^\beta}{k_B T} \int d^3p \, f^0(\vec{p}) \, v^\alpha v^\beta$$

$$= \frac{\tau e^{-i\omega t}}{1-i\omega t} \cdot \frac{nF^\alpha}{3k_BT} \int d^3v\, P(\vec{v})\, \vec{v}^2$$

and we arrive at

$$j^\alpha(t) = \frac{n\tau}{m} \cdot \frac{F^\alpha e^{-i\omega t}}{1-i\omega\tau}$$

For electrical conductivity, $j_e^\alpha = -e j^\alpha$ and $F^\alpha = -eE^\alpha$, so

$$j_e^\alpha(t) = \frac{ne^2\tau}{m} \frac{E^\alpha e^{-i\omega t}}{1-i\omega t} = \sigma_{\alpha\beta}(\omega)\, E^\beta e^{-i\omega t}$$

where the <u>frequency-dependent conductivity</u> is

$$\sigma_{\alpha\beta}(\omega) = \frac{ne^2\tau}{m} \cdot \frac{1}{1-i\omega\tau} \cdot \delta_{\alpha\beta}$$

For electrons we should use $f^0(\vec{p}) = \dfrac{h^{-3}}{e^{\beta(\varepsilon(\vec{p})-\mu)}-1}$

This affects the relation $n = n(\mu)$, but the above <u>Drude formula</u> remains unchanged.