Weakly Inhomogeneous Gas

With $f^0(\vec{p}) = C \exp\left(\frac{\mu + \vec{V}\cdot\vec{p} - \varepsilon(\vec{p})}{k_B T}\right)$ we allow $\mu(\vec{r},t)$, $V(\vec{r},t)$, and $T(\vec{r},t)$ to vary slowly in space and time, resulting in the distribution

$$f^0(\vec{r},\vec{p},t) = C \exp\left(\frac{\mu(\vec{r},t) + \vec{V}(\vec{r},t)\cdot\vec{p} - \varepsilon(\vec{p})}{k_B T(\vec{r},t)}\right)$$

This is not a sol$^{\underline{n}}$ to the BE. To solve the BE, we write $f(\vec{r},\vec{p},t) = f^0(\vec{r},\vec{p},t) + \delta f(\vec{r},\vec{p},t)$ and solve for $\delta f(\vec{r},\vec{p},t)$. So we have

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\cdot\frac{\partial}{\partial \vec{r}} + \vec{F}\cdot\frac{\partial}{\partial \vec{p}}\right)(f^0 + \delta f) = \left(\frac{df}{dt}\right)_{coll}$$

We need the chain rule to compute the $\vec{r}$, $\vec{p}$, and t derivatives of $f^0(\vec{r},\vec{p},t)$. The differential of f^0 is

$$df^0 = \left(d\mu + \vec{p}\cdot d\vec{V} + (\varepsilon - \vec{V}\cdot\vec{p} - \mu)\frac{dT}{T} - d\varepsilon\right)\left(-\frac{\partial f^0}{\partial \varepsilon}\right)$$

We'll skip the rest of the somewhat tedious derivation from § 8.6 of the lecture notes and just quote the result for NR ideal gases:

$$\frac{\partial \delta f}{\partial t} + \left\{ \frac{\varepsilon - h}{T} \vec{v} \cdot \vec{\nabla} T + m v_\alpha v_\beta Q_{\alpha\beta} - \frac{k_B \varepsilon}{c_V} \vec{\nabla} \cdot \vec{V} - \vec{F} \cdot \vec{v} \right\} \frac{f^0}{k_B T} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$$

where $h = c_p T$ = IG enthalpy per particle

$\vec{V}$ = local velocity

$Q_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} \right)$

$\vec{F}$ = external force

c_p, c_V = heat capacities per particle

IG $\Rightarrow h = c_p T$ (trans/rot only)

The RHS of the BE is the <u>collision integral</u>.

<u>Relaxation time approximation</u>: $\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{\delta f}{\tau}$

relaxation time

This is a considerable simplification, but it will yield much useful physics. We computed the RT to be $\tau = \frac{1}{n\bar{v}_{rel}\sigma} = \frac{\sqrt{\pi}}{4n\sigma}\left(\frac{m}{k_B T}\right)^{1/2}$ with σ = total 2-body scattering cross section and n being the particle number density.

Thermal Conductivity

Take $\vec{V} = 0$, $\vec{F} = 0$, and $\vec{\nabla}T$ constant. Then δf will be time-independent and we have

$$\delta f = -\frac{\tau(\varepsilon - c_p T)}{k_B T^2}(\vec{v}\cdot\vec{\nabla}T)f^0$$

The energy current is then

$$j_\varepsilon^\alpha = \int d^3p\, \varepsilon\, v^\alpha \delta f = -\frac{n\tau}{k_B T^2}\langle v^\alpha v^\beta \varepsilon(\varepsilon - c_p T)\rangle \frac{\partial T}{\partial x^\beta}$$

where the average $\langle \phi \rangle$ is with respect to the Maxwell dist^n:

$$\langle\phi\rangle = \int d^3p\, \phi\, f^0(\vec{p}) \Big/ \int d^3p\, f^0(\vec{p})$$
$$= \int d^3v\, P(\vec{v})\, \phi \quad ; \quad P(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-m\vec{v}^2/2k_BT}$$

Here T is the avg temperature since $\vec{j}_\varepsilon \propto \vec{\nabla} T$.

If $\phi = \phi(\varepsilon)$, we can use $P(\vec{v})\,d^3v = \mathcal{P}(\varepsilon)\,d\varepsilon$ with $\mathcal{P}(\varepsilon) = \frac{2}{\sqrt{\pi}}(k_BT)^{-3/2}\,\varepsilon^{1/2}\,e^{-\varepsilon/k_BT}$. Then find $\langle\varepsilon^\theta\rangle = \frac{2}{\sqrt{\pi}}\Gamma(\theta+\frac{3}{2})(k_BT)^\theta$. Thus

$$\vec{j}_\varepsilon = -\kappa\,\vec{\nabla}T \quad , \text{ with thermal conductivity}$$

$$\kappa = \frac{2n\tau}{3mk_BT^2}\langle\varepsilon^2(\varepsilon - \tfrac{5}{2}k_BT)\rangle = \frac{5n\tau k_B^2 T}{2m} = \frac{\pi}{8}\,n\ell\bar{v}\,c_p$$

Viscosity

Consider the situation depicted in the figure below. A fluid between two plates is sheared by applying a horizontal force to the upper plate. The lower plate is held fixed.