$$d\sigma = \frac{w(\vec{p}'|\vec{p})}{n_s|\vec{v}|}\, d^3p' \quad ; \quad \vec{v}(\vec{p}) = \frac{\partial \varepsilon}{\partial \vec{p}} = \frac{\vec{p}}{m}$$

n_s ← density of scatterers

and

$$\left(\frac{df}{dt}\right)_{coll} = \int d^3p' \left\{ w(\vec{p}|\vec{p}')\, f(\vec{r},\vec{p}',t) - w(\vec{p}'|\vec{p})\, f(\vec{r},\vec{p},t) \right\}$$

This is called <u>single particle scattering</u>. We also have <u>two particle scattering</u>. Let

$$w(\vec{p}',\vec{p}_1'\,|\,\vec{p},\vec{p}_1)\, f(\vec{p})\, f(\vec{p}_1)\, d^3p\, d^3p_1\, d^3p'\, d^3p_1' =$$

rate per $(\text{volume})^2$ to scatter two particles

$$\{|\vec{p}+d\vec{p},\ \vec{p}_1+d\vec{p}_1\rangle\} \longrightarrow \{|\vec{p}'+d\vec{p}',\ \vec{p}_1'+d\vec{p}_1'\rangle\}$$

Then

$$\left(\frac{df}{dt}\right)_{coll} = \int d^3p_1 \int d^3p' \int d^3p_1' \Big\{ w(\vec{p},\vec{p}_1\,|\,\vec{p}',\vec{p}_1')\, f_2(\vec{r},\vec{p}';\vec{r},\vec{p}_1';t)$$
$$- w(\vec{p}',\vec{p}_1'\,|\,\vec{p},\vec{p}_1)\, f_2(\vec{r},\vec{p};\vec{r},\vec{p}_1;t) \Big\}$$

Approximation:

$$f_2(\vec{r},\vec{p};\vec{r},\vec{p}_1;t) \simeq f(\vec{r},\vec{p};t)\, f(\vec{r},\vec{p}_1;t)$$

Then we have

$$\left(\frac{df}{dt}\right)_{coll} = \int d^3p_1 \int d^3p' \int d^3p_1' \Big\{ w(\vec{p}, \vec{p}_1 | \vec{p}', \vec{p}_1') f(\vec{r}, \vec{p}'; t) f(\vec{r}, \vec{p}_1'; t)$$
$$- w(\vec{p}', \vec{p}_1' | \vec{p}, \vec{p}_1) f(\vec{r}, \vec{p}; t) f(\vec{r}, \vec{p}_1; t) \Big\}$$

Detailed Balance

DB is a constraint on the equilibrium distⁿ $f^0(\vec{p})$.

For one-body scattering, the integrand of the collision term is

$$w(\vec{p}|\vec{p}') f(\vec{r}, \vec{p}', t) - w(\vec{p}'|\vec{p}) f(\vec{r}, \vec{p}, t)$$

Thus, a uniform and time-independent $f^0(\vec{p})$ satisfying

$$w(\vec{p}|\vec{p}') f^0(\vec{p}') = w(\vec{p}'|\vec{p}) f^0(\vec{p})$$

$$\Rightarrow \frac{f^0(\vec{p}')}{f^0(\vec{p})} = \frac{w(\vec{p}'|\vec{p})}{w(\vec{p}|\vec{p}')}$$

will make the integrand of $\left(\frac{df}{dt}\right)_{coll}$ vanish for all $\vec{p}$ and $\vec{p}'$. Thus $f^0(\vec{p})$ is a solⁿ to the BE with collisions. For two-body scattering, DB requires

$$w(\vec{p},\vec{p}_1\,|\,\vec{p}',\vec{p}_1')\,f^0(\vec{p}')\,f^0(\vec{p}_1') = w(\vec{p}',\vec{p}_1'\,|\,\vec{p},\vec{p}_1)\,f^0(\vec{p})\,f^0(\vec{p}_1)$$

Boltzmann's $\mathcal{H}$-theorem

Although the microscopic laws governing a system may be time-reversal invariant, the Boltzmann equation itself establishes an arrow of time.

Define

$$h(\vec{r},t) = \int d^3p\; f(\vec{r},\vec{p},t)\,\log\left[f(\vec{r},\vec{p},t)\,a^3\right]$$

$$\vec{j}(\vec{r},t) = \int d^3p\; f(\vec{r},\vec{p},t)\,\log\left[f(\vec{r},\vec{p},t)\,a^3\right]\frac{d\vec{r}}{dt}$$

where a is any constant with dimensions of action. After some grinding (see §8.3.7 in the lecture notes), we arrive at

$$\frac{\partial h(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{j}(\vec{r},t) = \sigma(\vec{r},t) < 0$$

with

$$\sigma(\vec{r},t) = -\frac{1}{2}\int d^3p \int d^3p_1 \int d^3p' \int d^3p_1' \ \{ w(\vec{p}\,',\vec{p}_1\,'|\vec{p},\vec{p}_1)$$

$$\times f(\vec{r},\vec{p}\,',t)\, f(\vec{r},\vec{p}_1\,',t)\,(x\log x - x + 1)$$

where

$$x(\vec{p},\vec{p}_1,\vec{p}\,',\vec{p}_1\,';\vec{r},t) = \frac{f(\vec{r},\vec{p},t)\, f(\vec{r},\vec{p}_1,t)}{f(\vec{r},p',t)\, f(\vec{r},\vec{p}_1\,',t)}$$

Integrating,

$$\frac{d\mathcal{H}(t)}{dt} = \int d^3r\, \sigma(\vec{r},t) < 0$$

$$\mathcal{H}(t) = \int d^3r\, h(\vec{r},t)$$

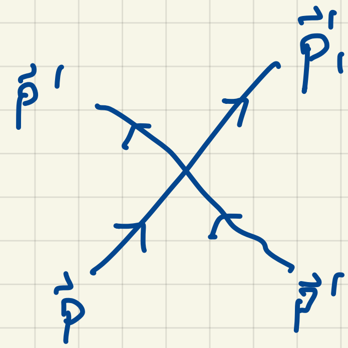
So $\mathcal{H}(t)$ always decreases -- the arrow of time!

To make $\sigma = 0$ we set $\chi = 1$ when $f(\vec{r}, \vec{p}, t) = f^0(\vec{p})$.

Then $f^0(\vec{p}) f^0(\vec{p}_1) = f^0(\vec{p}\,') f^0(\vec{p}_1\,')$, i.e.

$$\log f^0(\vec{p}) + \log f^0(\vec{p}_1) = \log f^0(\vec{p}\,') + \log f^0(\vec{p}_1\,')$$

But this means that $\log f^0(\vec{p})$ is a collisional invariant! For point particles there are five such CIs:



$$1, \ p^x, \ p^y, \ p^z, \ \varepsilon(\vec{p})$$

Thus we take

$$-\log f^0(\vec{p}) = \frac{\varepsilon(\vec{p}) - \vec{V} \cdot \vec{p} - \mu}{k_B T}$$

This is a Boltzmann dist$^{\underline{n}}$ in a moving frame.