# CBE

Suppose $\vec{F}_{ext} = -\vec{\nabla} U = 0$. We can then take $U = 0$, in which case CBE is

$$\frac{\partial f}{\partial t} + \vec{v}(\vec{p}) \cdot \frac{\partial f}{\partial \vec{r}} = 0$$

Solution (check it!): $f(\vec{r}, \vec{p}, t) = \varphi(\vec{r} - \vec{v}(\vec{p})t, \vec{p})$

where $f(\vec{r}, \vec{p}, t=0) = \varphi(\vec{r}, \vec{p})$ is any function.

Example: $\varphi(\vec{r}, \vec{p}) = C\, e^{-\vec{r}^2/2\sigma^2} e^{-\vec{p}^2/2K^2}$. Then

$$f(\vec{r}, \vec{p}, t) = C\, e^{-(\vec{r} - \frac{\vec{p}t}{m})^2/2\sigma^2} e^{-\vec{p}^2/2K^2}$$

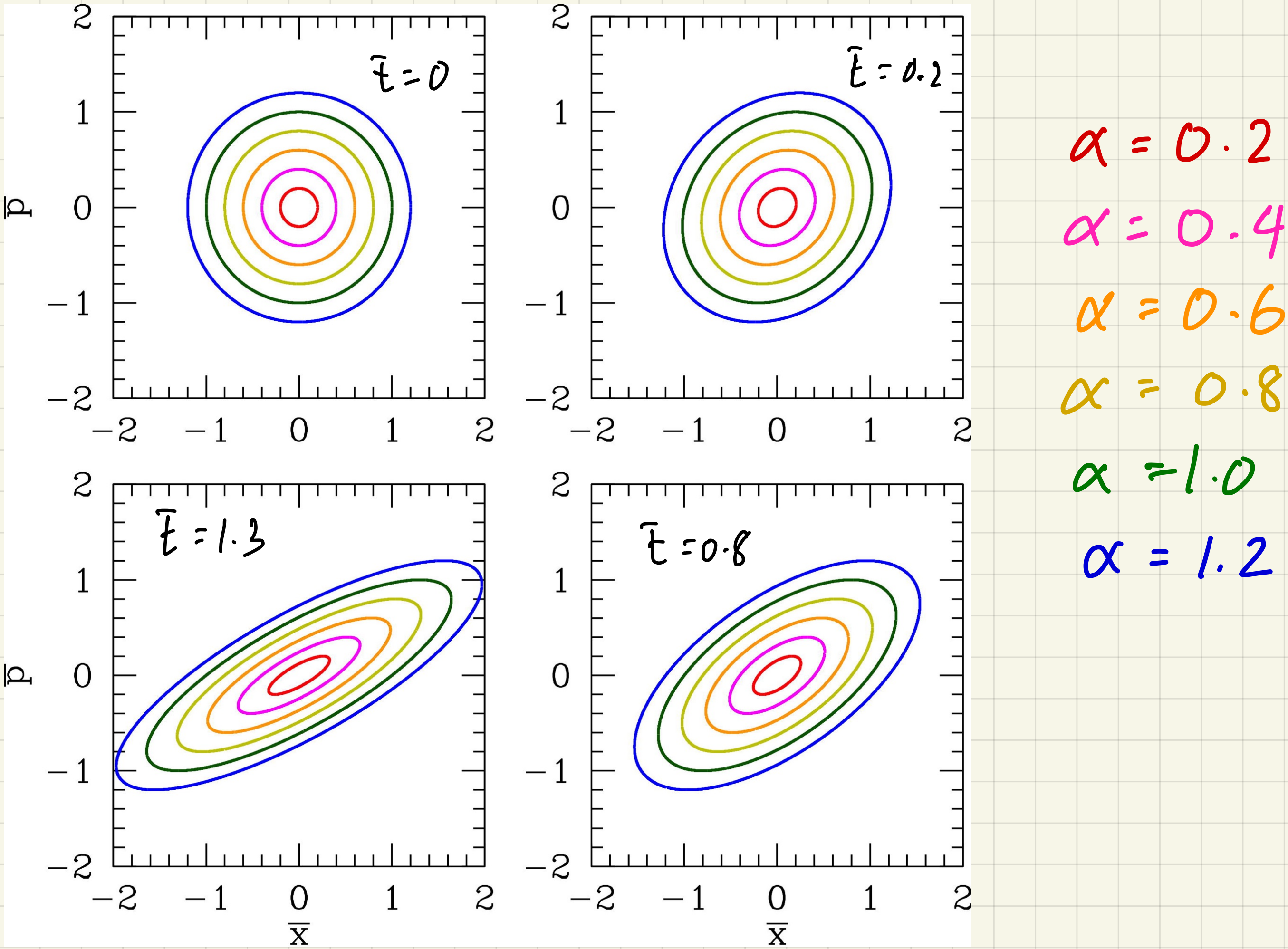
One-dim$^l$ version (rescaled $x, p, t$):

$$f(\bar{x}, \bar{p}, \bar{t}) = C e^{-(\bar{x} - \bar{p}\bar{t})^2/2} e^{-\bar{p}^2/2}$$

Level sets of $f$: set $f(\bar{x}, \bar{p}, \bar{t}) = C e^{-\alpha^2/2}$

Then $\bar{x} = \bar{p}\bar{t} \pm \sqrt{\alpha^2 - \bar{p}^2}$ are the eqns for the level sets. With constant force (rescaled):

$$f(\bar{x}, \bar{p}, \bar{t}) = C\, \varphi(\bar{x} - \bar{p}\bar{t} - \tfrac{1}{2}\bar{F}\bar{t}^2, \bar{p} - \bar{F}\bar{t})$$




## Collisional Invariants

Consider $A(\vec{r}, \vec{p})$. Then its avg value at time $t$ is

$$\langle A(t) \rangle = \int d^3r \int d^3p \; A(\vec{r}, \vec{p}) \, f(\vec{r}, \vec{p}, t)$$

$$\frac{d\langle A(t)\rangle}{dt} = \int d^3r \int d^3p \; A(\vec{r}, \vec{p}) \left\{ -\frac{\partial}{\partial \vec{r}} \cdot (\dot{\vec{r}} f) - \frac{\partial}{\partial \vec{p}} \cdot (\dot{\vec{p}} f) + \left(\frac{df}{dt}\right)_{coll} \right\}$$

Integrating by parts,

$$\frac{d\langle A\rangle}{dt} = \int d^3r \int d^3p \left\{ \left( \frac{\partial A}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} + \frac{\partial A}{\partial \vec{p}} \cdot \frac{d\vec{p}}{dt} \right) f + A(\vec{r},\vec{p}) \left( \frac{\partial f}{\partial t} \right)_{coll} \right\}$$

If $A(\vec{r}(t), \vec{p}(t))$ is conserved between collisions, then

$$\frac{d\langle A(t)\rangle}{dt} = \int d^3r \int d^3p \, A(\vec{r},\vec{p}) \left( \frac{\partial f}{\partial t} \right)_{coll}$$

A quantity $A(\vec{r},\vec{p})$ which is conserved in collisions is called a <u>collisional invariant</u>. Examples include particle number ($A=1$), total momentum ($A = p^\alpha$), and total kinetic energy ($A = \varepsilon(\vec{p})$).

## <u>Scattering Processes</u>

Let $w(\vec{p}'|\vec{p}) f(\vec{p}) \, d^3p \, d^3p' = \begin{cases} \text{rate per volume to scatter} \\ \{|\vec{p}+d\vec{p}\rangle\} \rightarrow \{|\vec{p}'+d\vec{p}'\rangle\} \\ \text{at time } t \end{cases}$

Units of $w \, d^3p$ are $T^{-1}$ because $[f(\vec{p}) d^3p] = L^{-3}$. The differential scattering cross section is then

$$d\sigma = \frac{w(\vec{p}\,'|\vec{p})}{n_s|\vec{v}|}\,d^3p' \quad ; \quad \vec{v}(\vec{p}) = \frac{\partial \varepsilon}{\partial \vec{p}} = \frac{\vec{p}}{m}$$

density of scatterers

and

$$\left(\frac{df}{dt}\right)_{coll} = \int d^3p' \left\{ w(\vec{p}|\vec{p}\,')\, f(\vec{r},\vec{p}\,',t) - w(\vec{p}\,'|\vec{p})\, f(\vec{r},\vec{p},t) \right\}$$

This is called <u>single particle scattering</u>. We also have <u>two particle scattering</u>. Let

$$w(\vec{p}\,',\vec{p}_1\,'|\vec{p},\vec{p}_1)\, f(\vec{p})\, f(\vec{p}_1)\, d^3p\, d^3p_1\, d^3p'\, d^3p_1' =$$

rate per $(\text{volume})^2$ to scatter two particles $\{|\vec{p}+d\vec{p}, \vec{p}_1+d\vec{p}_1\rangle\} \longrightarrow \{|\vec{p}\,'+d\vec{p}\,', \vec{p}_1\,'+d\vec{p}_1\,'\rangle\}$

Then

$$\left(\frac{df}{dt}\right)_{coll} = \int d^3p_1 \int d^3p' \int d^3p_1' \left\{ w(\vec{p},\vec{p}_1|\vec{p}\,',\vec{p}_1\,')\, f_2(\vec{r},\vec{p}\,';\vec{r},\vec{p}_1\,';t) - w(\vec{p}\,',\vec{p}_1\,'|\vec{p},\vec{p}_1)\, f_2(\vec{r},\vec{p};\vec{r},\vec{p}_1;t) \right\}$$

Approximation:

$$f_2(\vec{r},\vec{p};\vec{r},\vec{p}_1;t) \approx f(\vec{r},\vec{p};t)\, f(\vec{r},\vec{p}_1;t)$$

Then we have

$$\left(\frac{df}{dt}\right)_{coll} = \int d^3p_1 \int d^3p' \int d^3p_1' \Big\{ w(\vec{p}, \vec{p}_1 | \vec{p}', \vec{p}_1') f(\vec{r}, \vec{p}'; t) f(\vec{r}, \vec{p}_1'; t)$$
$$- w(\vec{p}', \vec{p}_1' | \vec{p}, \vec{p}_1) f(\vec{r}, \vec{p}; t) f(\vec{r}, \vec{p}_1; t) \Big\}$$

## Detailed Balance

DB is a constraint on the equilibrium dist$^n$ $f^0(\vec{p})$.

For one-body scattering, the integrand of the collision term is

$$w(\vec{p}|\vec{p}') f(\vec{r}, \vec{p}', t) - w(\vec{p}'|\vec{p}) f(\vec{r}, \vec{p}, t)$$

Thus, a uniform and time-independent $f^0(\vec{p})$ satisfying

$$w(\vec{p}|\vec{p}') f^0(\vec{p}') = w(\vec{p}'|\vec{p}) f^0(\vec{p})$$

$$\Rightarrow \frac{f^0(\vec{p}')}{f^0(\vec{p})} = \frac{w(\vec{p}'|\vec{p})}{w(\vec{p}|\vec{p}')}$$

will make the integrand of $\left(\frac{df}{dt}\right)_{coll}$ vanish for all $\vec{p}$ and $\vec{p}'$. Thus $f^0(\vec{p})$ is a sol$^n$ to the BE with collisions. For two-body scattering, DB requires

$$w(\mathbf{p},\vec{p}_1\,|\,\vec{p}',\vec{p}_1')\,f(\vec{r},\vec{p}';t)\,f(\vec{r},\vec{p}_1';t)$$
$$= w(\vec{p}',\vec{p}_1'\,|\,\mathbf{p},\vec{p}_1)\,f(\vec{r},\vec{p};t)\,f(\vec{r},\vec{p}_1;t)$$

## Boltzmann's $\mathcal{H}$-theorem

Although the microscopic laws governing a system may be time-reversal invariant, the Boltzmann equation itself establishes an arrow of time.
Define

$$h(\vec{r},t) = \int d^3p\; f(\vec{r},\vec{p},t)\,\log\left[f(\vec{r},\vec{p},t)\,a^3\right]$$
$$\vec{j}(\vec{r},t) = \int d^3p\; f(\vec{r},\vec{p},t)\,\log\left[f(\vec{r},\vec{p},t)\,a^3\right]\frac{d\vec{r}}{dt}$$

where $a$ is any constant with dimensions of action. After some grinding (see §8.3.7 in the lecture notes), we arrive at

$$\frac{\partial h(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{j}(\vec{r},t) = \sigma(\vec{r},t) < 0$$

with

$$\sigma(\vec{r},t) = -\frac{1}{2}\int d^3p \int d^3p_1 \int d^3p' \int d^3p_1' \left\{ w(\vec{p}', \vec{p}_1' | \vec{p}, \vec{p}_1) \right.$$

$$\left. \times f(\vec{r},\vec{p}',t)\, f(\vec{r},\vec{p}_1',t)\, (x \log x - x + 1) \right)$$

where

$$x(\vec{p},\vec{p}_1,\vec{p}',\vec{p}_1'; \vec{r},t) = \frac{f(\vec{r},\vec{p},t)\, f(\vec{r},\vec{p}_1,t)}{f(\vec{r},\vec{p}',t)\, f(\vec{r},\vec{p}_1',t)}$$

Integrating,

$$\frac{d\mathcal{H}(t)}{dt} = \int d^3r\, \sigma(\vec{r},t) < 0$$

$$\mathcal{H}(t) = \int d^3r\, h(\vec{r},t)$$

So $\mathcal{H}(t)$ always decreases -- the arrow of time!