Transport

In the equilibrium states we've studied, there is no <u>current</u> flowing. Particles themselves are whizzing around in, e.g., an ideal gas, but there is no net <u>transport</u> of particle number across the entire system. Let

$$f(\vec{r},\vec{p},t)\,d^3r\,d^3p = \text{\# particles with positions within } d^3r \text{ of } \vec{r} \text{ and momenta within } d^3p \text{ of } \vec{p} \text{ at time } t$$

In equilibrium,

number density

$$f^0(\vec{r},\vec{p},t) = f^0(\vec{p}) = \frac{n}{(2\pi m k_B T)^{3/2}}\, e^{-\vec{p}^{\,2}/2mk_BT}$$

This is the Maxwell dist$^{\underline{n}}$. The local number density is $n(\vec{r},t) = \int d^3p\, f(\vec{r},\vec{p},t)$. Thus in equilibrium we have $n(\vec{r},t) = n$, a constant.

Currents

The particle number current is given by

$$\vec{j}(\vec{r},t) = \int d^3p \, f(\vec{r},\vec{p},t)\, \vec{v}(\vec{p})$$

velocity $\vec{v} = \frac{\partial \varepsilon}{\partial \vec{p}}$

The energy current is

energy

$$\vec{j}_\varepsilon(\vec{r},t) = \int d^3p \, f(\vec{r},\vec{p},t)\, \varepsilon(\vec{p})\, \vec{v}(\vec{p})$$

For ballistic particles, $\varepsilon(\vec{p}) = \frac{\vec{p}^{\,2}}{2m}$, $\vec{v}(\vec{p}) = \frac{\vec{p}}{m}$.

Thermodynamics says $đq = T\,ds = d\mathcal{E} - \mu\, dn$ where $s, \mathcal{E}, n$ are entropy, energy, and number densities, assuming local equilibrium. (Don't confuse $\mathcal{E}$ and $\varepsilon(\vec{p})$.) We define the heat current,

$$\vec{j}_q(\vec{r},t) = \vec{j}_\varepsilon(\vec{r},t) - \mu\, \vec{j}(\vec{r},t)$$

and the entropy current,

$$\vec{j}_s(\vec{r},t) = \frac{1}{T}\, \vec{j}_q(\vec{r},t)$$

If $f(\vec{r},\vec{p},t) = f^0(\vec{p})$, then $\vec{j} = \vec{j}_\varepsilon = \vec{j}_q = \vec{j}_s = 0$.

No current flows under equilibrium conditions.

Transport coefficients

Suppose the temperature $T(\vec{r})$ is spatially varying. This means that a <u>local equilibrium</u> is established everywhere in space. This results in the presence of currents which seek to establish a <u>global</u> eqbm. Particles adjust to the local equilibria through scattering processes. Consider some intensive quantity whose local thermodynamic value is $\phi(\vec{r})$. For example, we might take $\phi(\vec{r}) = \langle \varepsilon \rangle_{\vec{r}}$. Now let's compute the ϕ-current, $\vec{j}_\phi$. For simplicity, we assume $\phi = \phi(z)$. Then

$$\vec{j}_\phi = n\hat{z}\int_{v_z>0} d^3v\, P(\vec{v})\, v_z\, \phi(z - \ell\cos\theta) + n\hat{z}\int_{v_z<0} d^3v\, P(\vec{v})\, v_z\, \phi(z + \ell\cos\theta)$$

ϕ-current across $z=0$ surface

Here

$$P(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-m\vec{v}^2/2k_BT}$$

is the Maxwell velocity dist^{n} and ℓ is the <u>mean free</u> path, which is the average distance between collisions. Note $\cos\theta = v_z/v$. Expanding $\phi(z \pm \ell\cos\theta) = \phi(z) \pm \ell \frac{v_z}{v} \phi'(z) + \ldots$ and thus

$$\vec{j}_\phi = -n\ell \frac{\partial\phi}{\partial z} \hat{z} \int d^3v\, P(\vec{v}) \frac{v_z^2}{v} = -\frac{1}{3} n \ell \bar{v} \frac{\partial \phi}{\partial z} \hat{z}$$

with $\bar{v} = (8k_BT/\pi m)^{1/2}$ is the avg particle speed. If $\phi(z) = \phi(T(z))$, then

$$\vec{j}_\phi = -\frac{1}{3} n \ell \bar{v} \frac{\partial\phi}{\partial T} \vec{\nabla} T \equiv -K \vec{\nabla} T$$

where
$$K = \frac{1}{3} n \ell \bar{v} \frac{\partial \phi}{\partial T}$$

is a <u>transport coefficient.</u> If $\phi = \langle \varepsilon \rangle$, the average energy per particle, then

$$\left.\frac{\partial \phi}{\partial T}\right|_p = C_p$$ ← N.B. local eqbm at fixed p

Thus, $\vec{J}_E = -\kappa \vec{\nabla} T$ with $\kappa = \frac{1}{3} n \ell \bar{v} C_p \propto \sqrt{T}$.

We call κ the <u>thermal</u> <u>conductivity</u>.

Calculation of the mean free path

Avg relative speed of 2 particles:

$$\bar{v}_{rel} = \langle |\vec{v} - \vec{v}\,'| \rangle = \int d^3v \int d^3v'\, P(\vec{v})\, P(\vec{v}\,') = \frac{4}{\sqrt{\pi}} \left(\frac{k_B T}{m} \right)^{1/2}$$

Then set $n \sigma \bar{v}_{rel} \tau \equiv 1$ where

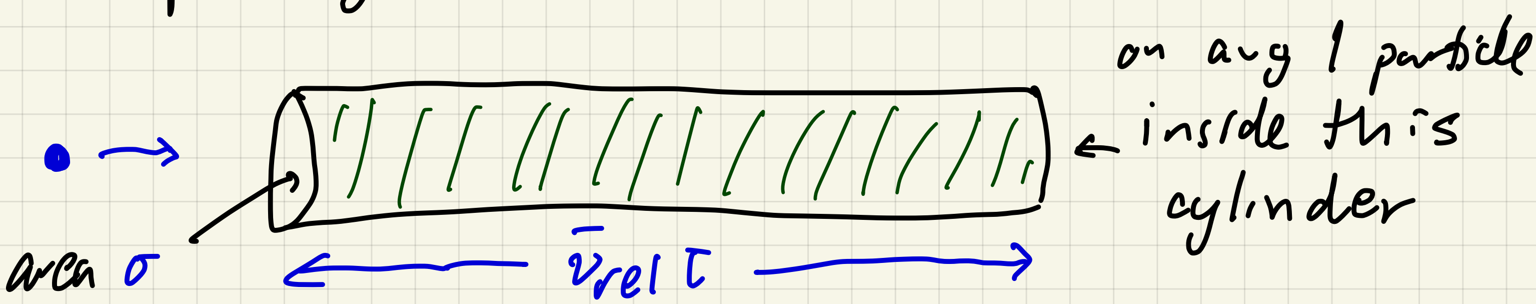
σ = total scattering cross section

τ = scattering time

Thus, $\frac{1}{\tau} = n \bar{v}_{rel} \sigma \Rightarrow \tau(T) = \frac{\sqrt{\pi}}{4 n \sigma} \left(\frac{m}{k_B T} \right)^{1/2} \propto T^{-1/2}$

The mean free path is $\ell = \bar{v} \tau = 1/\sqrt{2} n \sigma \propto T^0$

Graphic argument:

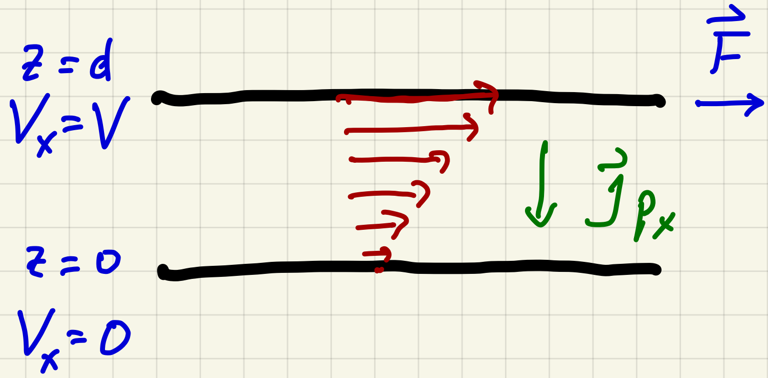


More generally,

$$P(\vec{r},\vec{v}) = n_0 \left(\frac{m}{2\pi k_B T(\vec{r})}\right)^{3/2} e^{-m(\vec{v}-\vec{V}(\vec{r}))^2/2k_B T(\vec{r})} e^{\mu(\vec{r})/k_B T(\vec{r})}$$

$$\Rightarrow n(\vec{r}) = n_0 \, e^{\mu(\vec{r})/k_B T(\vec{r})} \quad ; \quad N = n_0 \int d^3r \, e^{\mu(\vec{r})/k_B T(\vec{r})}$$

<u>Viscosity</u>



With $\phi = \langle p_x \rangle = m V_x(z)$

we obtain

$$J^z_{p_x} = -\frac{1}{3} n m \ell \bar{v} \frac{\partial V_x}{\partial z} = -\eta \frac{\partial V_x}{\partial z}$$

where $\eta = \frac{1}{3} n m \ell \bar{v}$ is the <u>shear viscosity</u>

The drag force on the top plate is

$$F_{drag} = -\eta A \frac{V}{d}$$

Units of shear viscosity : $[\eta] = \frac{M}{L \cdot T} = \frac{F \cdot T}{L^2}$

MKS : $[\eta] = \frac{N s}{m^2} = \frac{kg}{m \cdot s} = Pa \cdot s$

Note in solids

$$F_{shear} = -\underbrace{G}_{\text{shear modulus}} A \theta = -GA\frac{\Delta x}{d}$$