Sextic Potential

Now consider $f(m) = f_0 + \frac{1}{2}am^2 + \frac{1}{4}bm^4 + \frac{1}{6}cm^6$ with $c>0$ but a,b can be positive or negative. The eqn $\frac{\partial f}{\partial m} = (a+bm^2+cm^4)m = 0$ is quintic and thus has five roots in the complex m plane. Since the coefficients $a,b,c \in \mathbb{Z}$, if $f'(m)=0$ then $f'(m^*)=0$ where m^* is the complex conjugate of m. The five roots are:

$$m_1 = 0$$

$$m_2 = +\sqrt{-\frac{b}{2c} + \sqrt{\frac{b^2}{4c^2} - \frac{a}{c}}}$$

$$m_3 = +\sqrt{-\frac{b}{2c} - \sqrt{\frac{b^2}{4c^2} - \frac{a}{c}}}$$

$$m_4 = -\sqrt{-\frac{b}{2c} - \sqrt{\frac{b^2}{4c^2} - \frac{a}{c}}} = -m_3$$

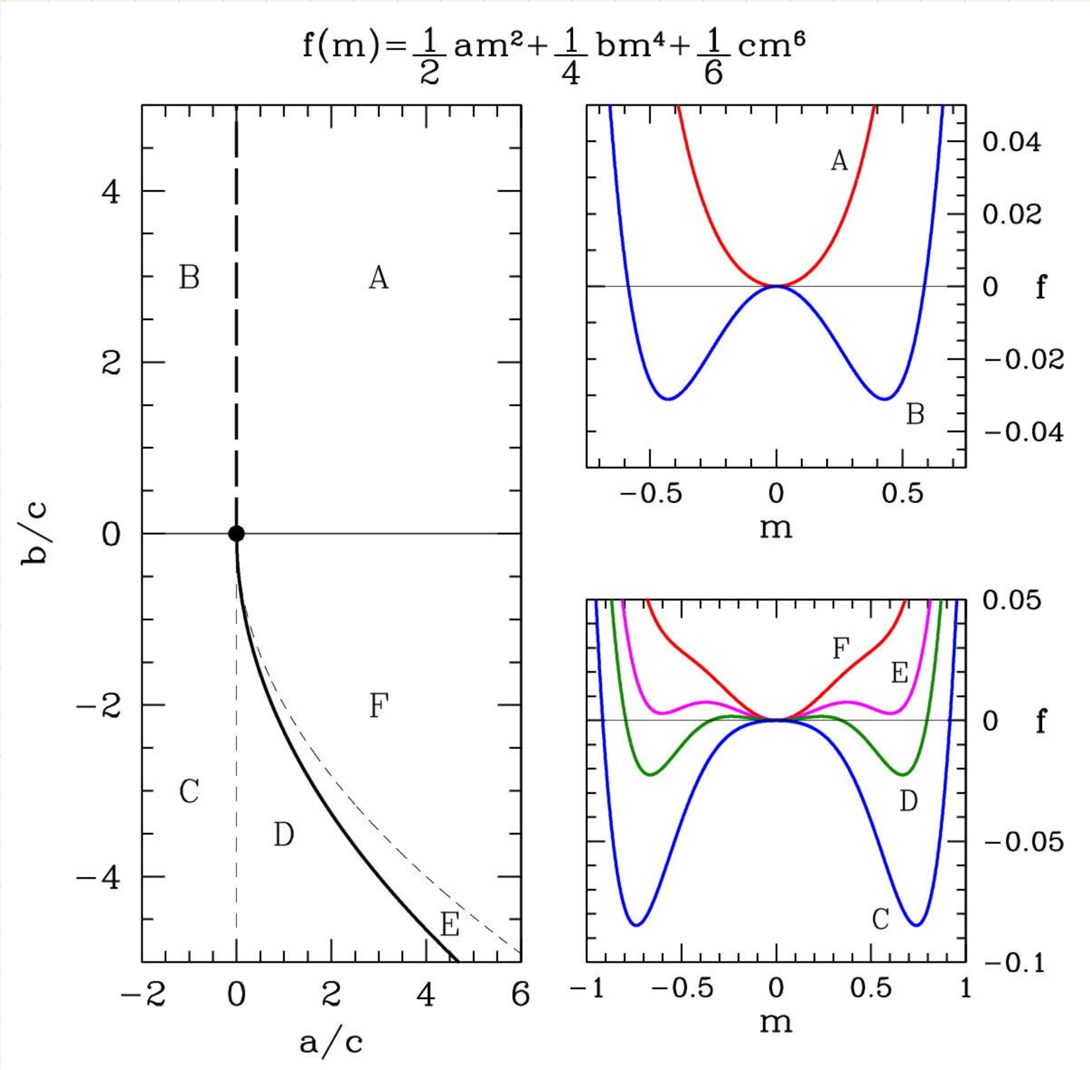
$$m_5 = -\sqrt{-\frac{b}{2c} + \sqrt{\frac{b^2}{4c^2} - \frac{a}{c}}} = -m_2$$

Note that the discriminant $\Delta \equiv \frac{b^2}{4c^2} - \frac{a}{c}$ is negative for $a > \frac{b^2}{4c} > 0$. In this case, $m_1 = 0$ is the only real root. This is the symmetric (under $\mathbb{Z}_2$) high-temperature phase. If $b > 0$, no additional roots appear until $a = 0$, where we encounter a supercritical pitchfork bifurcation and two new real roots appear at $m = \pm m_2$, which are degenerate minima of $f(m)$. The root $m = m_1 = 0$ is a local maximum, just as in the quartic case. (The roots at $m = \pm m_3$ are both pure imaginary.)

If $b < 0$, things are more interesting. At $a = a_{SN} = \frac{b^2}{4c}$ we encounter two simultaneous saddle node bifurcations occur at the solutions to

$$\left.\begin{aligned} f'(m) &= am + bm^3 + cm^5 = 0 \\ f''(m) &= a + 3bm^2 + 5cm^4 = 0 \end{aligned}\right\} \quad m^2 = \pm\sqrt{-\frac{b}{2c}}$$

These develop into local minima at $\pm m_2$ and local maxima at $\pm m_3$. Eventually the free energy at the local minima drop below zero. This occurs when $f(\pm m_2) = f(0) \Rightarrow a = \frac{3b^2}{16c}$. The order parameter then jumps discontinuously from $m=0$ to $m = \pm\left(-\frac{b}{4c}\right)^{1/2}$, heralding the entry into region D in the diagram below. As a



decreases further, there are no other phase transitions and the system evolves smoothly.

Under the dynamics $\dot{m} = -\Gamma \frac{\partial f}{\partial m}$, we can rescale to write $\dot{u} = -\varphi'(u)$ with

$\varphi(u) = \frac{1}{2} r u^2 + \frac{1}{4} \operatorname{sgn}(b)\, u^4 + \frac{1}{6} u^6$ and $r = \frac{ac}{b^2}$.

For $b>0$ the physics is as for the quartic case.

But for $b<0$ we get hysteresis: