Cubic terms in Landau theory

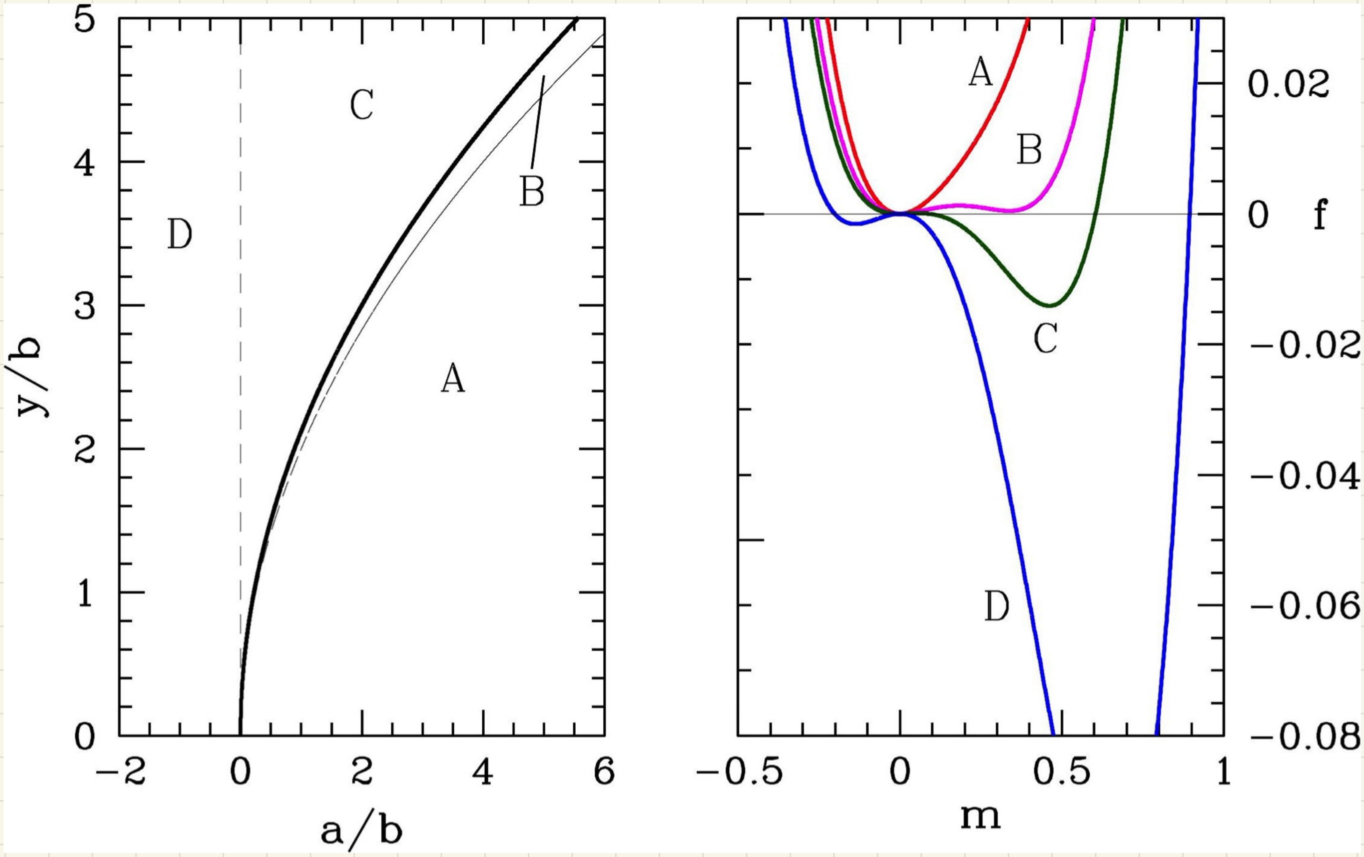
Let $f = f_0 + \frac{1}{2}am^2 - \frac{1}{3}ym^3 + \frac{1}{4}bm^4 \quad (b>0)$

$$f'(m) = (a - ym + bm^2)m \Rightarrow \begin{cases} m = 0 \\ m = m_\pm = \frac{y}{2b} \pm \sqrt{\left(\frac{y}{2b}\right)^2 - \frac{a}{b}} \end{cases}$$

Thus if $y^2 < 4ab$ the only sol^{n} is $m=0$. But if $y^2 > 4ab$ then we have 3 sol^{ns}. If $a<0$ then clearly $m=m_+$ is the global minimum. For $0 < a < \frac{y^2}{4b}$ there is a local minimum at $m=0$, a local maximum at $m_- > 0$, and a local minimum at $m=m_+$. Which is the global minimum, $m=0$ or $m=m_+$? Set $f(0) = f(m_+)$:

$$\left.\begin{aligned} a - ym + bm^2 &= 0 \\ \tfrac{1}{2}a - \tfrac{1}{3}ym + \tfrac{1}{4}bm &= 0 \end{aligned}\right\} \Rightarrow m_+ = \frac{3a}{y} \Rightarrow y^2 = \frac{9}{2}ab$$

We thus have the following picture:

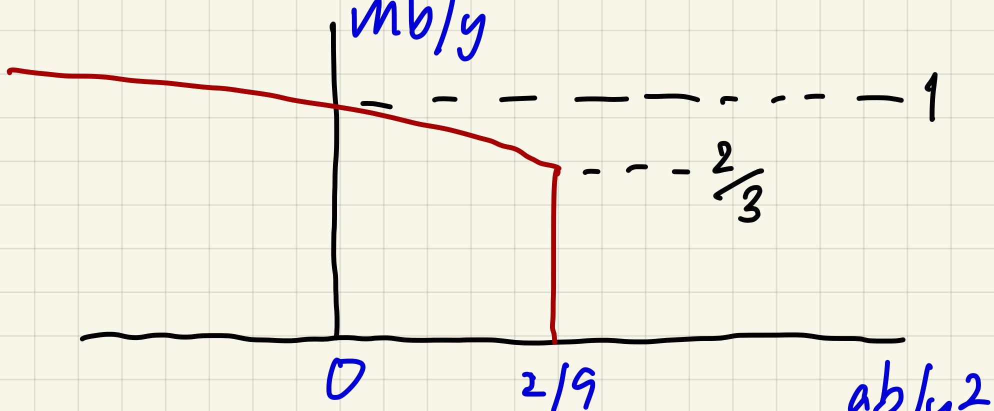


- $a < \frac{2y^2}{9b}$: 3 real roots to $f'(m) = 0$

  lowest energy at $m_+ = \frac{y}{2b} + \sqrt{\left(\frac{y}{2b}\right)^2 - \frac{a}{b}}$

- $\frac{2y^2}{9b} < a < \frac{y^2}{4b}$: 3 real roots, min at $m = 0$

- $\frac{y^2}{4b} < a$: 1 real root at $m = 0$



Thus, as a is decreased from large values, a first order transition at $a = \frac{2}{9}$ <u>preempts</u> the second order transition that would have taken place at $a = 0$. <u>Magnetization dynamics</u>: consider the model

$$\frac{dm}{dt} = -\Gamma \frac{\partial f}{\partial m} = -\Gamma\left(am - ym^2 + bm^3\right)$$

and let $m \equiv b^{-1}yu$, $a \equiv b^{-1}y^2r$, $t = b^2\tau/\Gamma y^3$ so

$$\frac{du}{d\tau} = \left(-r + u - u^2\right)u = -\frac{\partial \varphi}{\partial u}$$

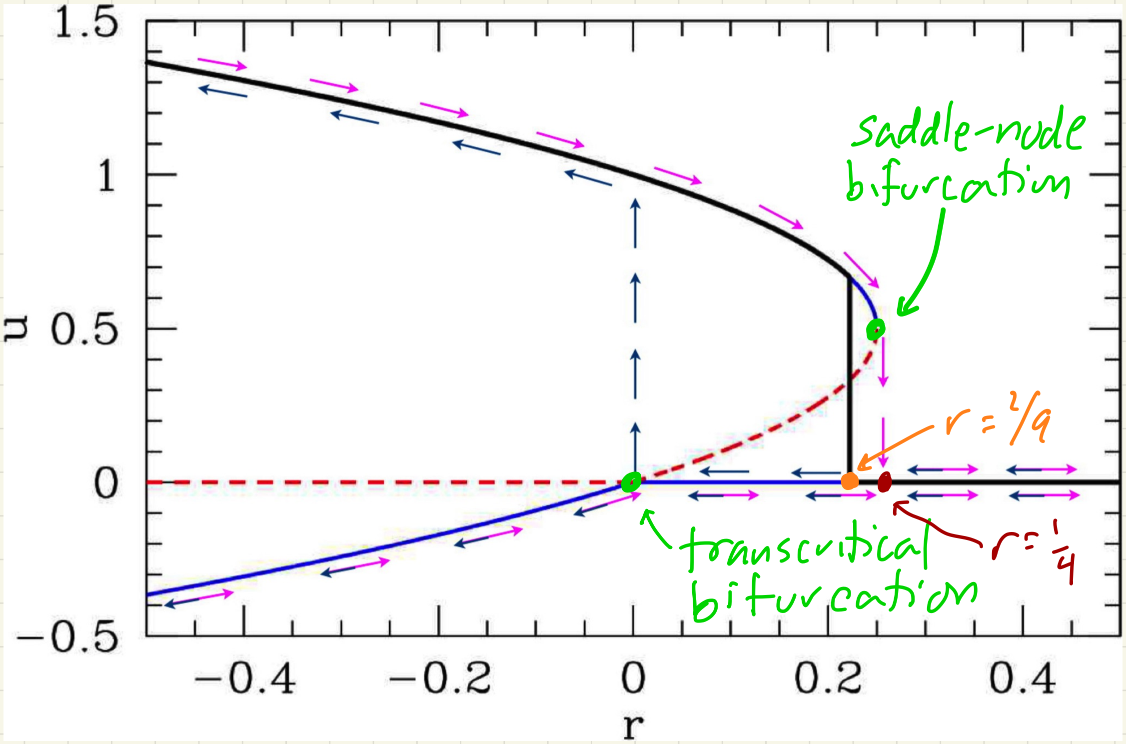
where $\varphi(u) = \frac{1}{2}ru^2 - \frac{1}{3}u^3 + \frac{1}{4}u^4 = b^{-3}y^4(f - f_0)$

is the dimensionless singular part of the free energy density (i.e. without the f_0 contribution).

Fixed points occur when $\varphi'(u) = 0 \Rightarrow$

- $u = 0$ (r-axis in the (r,u) plane)
- $r = u - u^2$ (left-opening parabola in (r,u) plane)

Slowly ramping the control parameter r up and down yields a hysteresis loop in the (r,u) plane:



Stable branches:
- $u = 0$ for $r > 0$
- $u = \frac{1}{2} - \sqrt{\frac{1}{4} - r}$ for $r < 0$
- $u = \frac{1}{2} + \sqrt{\frac{1}{4} - r}$ for $r < \frac{1}{4}$

Unstable branches:
- $u = 0$ for $r < 0$
- $u = \frac{1}{2} - \sqrt{\frac{1}{4} - r}$ for $0 < r < \frac{1}{4}$