Landau Theory of Phase Transitions

- Idea: express free energy $f(\vec{m})$ in terms of an <u>order parameter</u> $\vec{m}$ which vanishes in the high temperature disordered phase and is finite in low temperature ordered phases. $f(\vec{m})$ may be invariant under a group G of symmetry operations, such as $G = \mathbb{Z}_2$ acting on a scalar OP m, in which case $f(m) = f(\varepsilon m)$ where $\varepsilon \in \{1, -1\} \cong \mathbb{Z}_2$. A nonzero $\vec{m}$ in the ordered phase then breaks the symmetry, or, more precisely, the free energy $f(\vec{m})$ is symmetric but the minima are degenerate and G is <u>spontaneously broken</u> as the system picks a particular minimizing $\vec{m}$. The presence of an external field $\vec{h}$ <u>explicitly</u> breaks the G-symmetry.

Quartic, $\mathbb{Z}_2$ symmetric

Let $f(m,\theta) = f_0 + \frac{1}{2}am^2 + \frac{1}{4}bm^4$. Then

$$\frac{\partial f}{\partial m} = am + bm^3 = (a+bm^2)m = 0$$

Sol^{ns} : $a<0 \Rightarrow \underset{\text{max}}{m=0}, \ \underset{\text{min}}{m=\pm\sqrt{-a/b}}$

$a>0 \Rightarrow m=0$

Thus $f(\theta, a<0) = f_0 - \dfrac{a^2}{4b}$

$f(\theta, a>0) = f_0$

Here $a(\theta) \propto a_0(\theta-\theta_c)$ for $|\theta-\theta_c| \ll 1$

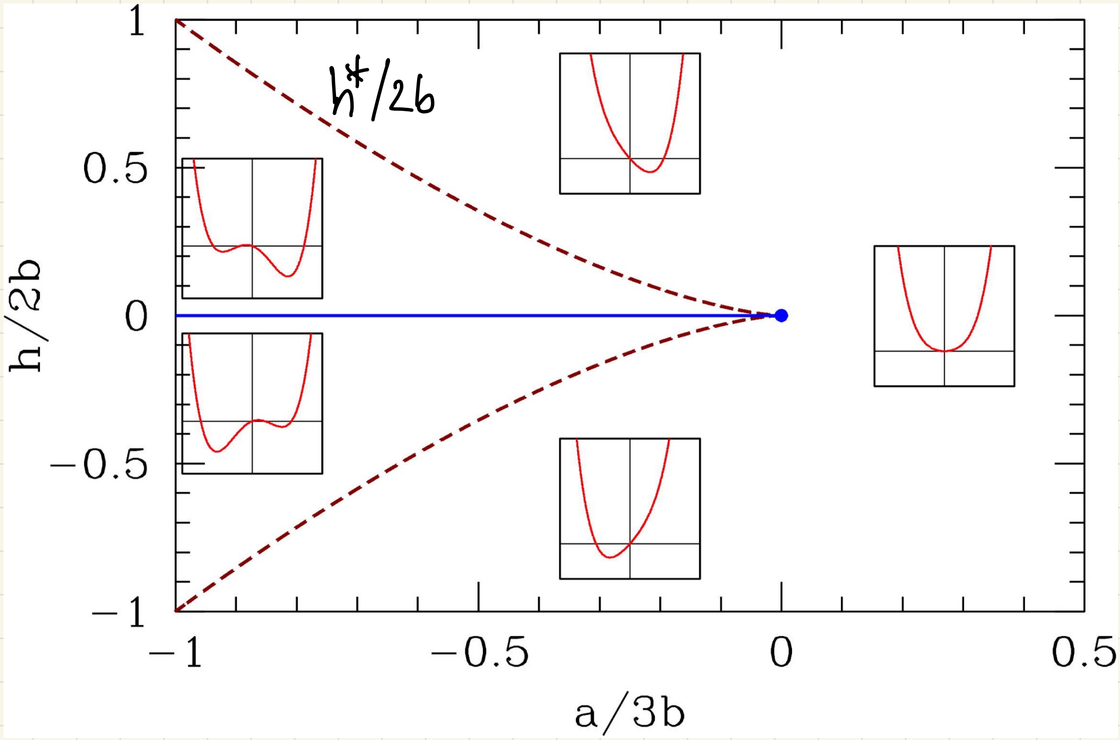
Then $\Delta c = c(a=0^+) - c(a=0^-)$

$$= -\theta \frac{\partial^2 f}{\partial \theta^2}\bigg|_{\theta_c} = -\frac{\theta_c[a'(\theta_c)]^2}{2b(\theta_c)}$$ discontinuity

Break $\mathbb{Z}_2$ with $h \neq 0$:

$$f(m,\theta,h) = f_0 + \frac{1}{2}am^2 + \frac{1}{4}bm^4 - hm$$

$$\frac{\partial f}{\partial m} = am + bm^3 - h = 0$$



Saddle-node bifurcation: $f'(m=0)$ and $f''(m)=0$

$$f''(m) = a + 3bm^2 = 0 \Rightarrow m^* = \pm\left(-\frac{a}{3b}\right)^{1/2}$$

$$f'(m^*) = 0 \Rightarrow h^* = a\left(-\frac{a}{3b}\right)^{1/2} + b\left(-\frac{a}{3b}\right)^{3/2}$$

$$h^*(a,b) = \pm\frac{2}{\sqrt{27}}\,\frac{(-a)^{3/2}}{b^{1/2}} \quad \text{for } a<0$$

Note $\frac{h^*}{2b} = \pm\left(-\frac{a}{3b}\right)^{3/2}$ (see figure)

Equivalently, $a^*(h,b) = -\frac{3}{2^{2/3}}\, b^{1/3}\, |h|^{2/3}$