Specific heat

For $h=0$ and $|\theta-\theta_c| \ll 1$, we have

$$f(\theta, h=0) = f_0 + \frac{1}{2} t m^2 + \frac{1}{12} m^4 + \dots \ ; \quad f_0 = -\theta \log 2$$

where $t \equiv (\theta-\theta_c)/\theta_c$ = "reduced temperature". Then

$$f(\theta, h=0) = \begin{cases} f_0 - \frac{3}{4}(\theta-\theta_c)^2 + \mathcal{O}((\theta-\theta_c)^4) \ , & \theta < \theta_c \\ f_0 \ , & \theta > \theta_c \end{cases}$$

and thus, recalling $C_V = T \frac{\partial S}{\partial T}\Big|_V = -T \frac{\partial^2 F}{\partial T^2}$

$$c_v = -\theta \frac{\partial^2 f}{\partial \theta^2} = \begin{cases} \frac{3}{2} \ , & \theta < \theta_c \\ 0 \ , & \theta > \theta_c \end{cases}$$

Hence there is a <u>discontinuity</u> in $C_v(\theta, h=0)$ at $\theta = \theta_c$. Writing $C_v(\theta, h=0) \propto |\theta-\theta_c|^{-\alpha}$, we have that the specific heat exponent is $\alpha = 0$.

Finite external field

Assume $|h| \ll |\theta-\theta_c| \ll 1$ and expand:

$$f(m,\theta,h) = f_0 + \frac{1}{2}(\theta-\theta_c) m^2 + \frac{1}{12} m^4 - hm + \dots$$

Thus $\dfrac{\partial f}{\partial m} = \frac{1}{3}m^3 + (\theta - \theta_c)m - h = 0$

For $\theta > \theta_c$, there is no spontaneous magnetization at $h=0$, so we can ignore the m^3 term, whence

$$m(\theta > \theta_c, h) = \frac{h}{\theta - \theta_c}$$

and the magnetic susceptibility is

$$\chi(\theta > \theta_c) = \frac{\partial m}{\partial h}\bigg|_{\substack{\theta > \theta_c \\ h=0}} = \frac{1}{\theta - \theta_c}$$

For $\theta < \theta_c$ we write $m = m_0 + \delta m$ with $m_0 = \pm\sqrt{3}\,(\theta_c - \theta)$ in which case we find

$$\frac{1}{3}\left(\cancel{m_0^3} + 3m_0^2\,\delta m + \dots\right) + (\theta - \theta_c)(\cancel{m_0} + \delta m) = h$$

$$\underbrace{\left[m_0^2 + (\theta - \theta_c)\right]}_{=2(\theta_c - \theta)}\delta m + \mathcal{O}(\delta m^2) = h \;\Rightarrow$$

$$m(\theta < \theta_c) = m_0(\theta) + \frac{h}{2(\theta_c - \theta)} \;\Rightarrow\; \chi(\theta < \theta_c) = \frac{1}{2(\theta_c - \theta)}$$

Thus we have

amplitudes, exponent

$$\chi(\theta) = A_\pm \, |\theta - \theta_c|^{-\gamma}$$

with $\gamma = 1$, $A_+ = 1$, $A_- = \frac{1}{2}$.

Magnetization dynamics

Phenomenological dynamics: $\dfrac{dm}{dt} = -\Gamma \dfrac{\partial f}{\partial m}$

where Γ is a rate. Dimless time $s \equiv \Gamma t$, so

$\dfrac{dm}{ds} = -\dfrac{\partial f}{\partial m}$. Thus

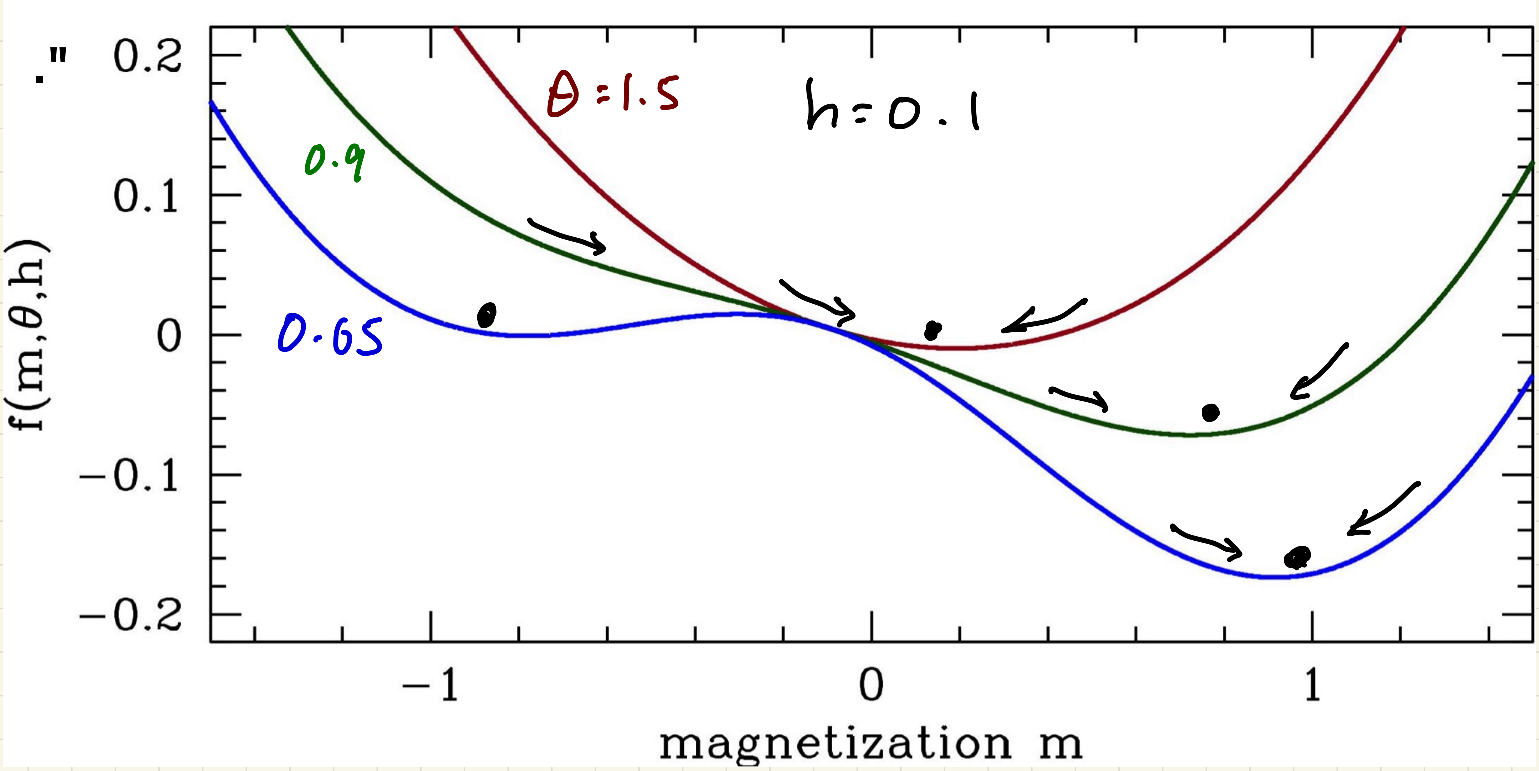
$$\frac{d}{ds} f(m(s)) = \frac{\partial f}{\partial m} \frac{dm}{ds} = -\left(\frac{\partial f}{\partial m}\right)^2 \leq 0$$

and thus f always flows downhill. The flow stops when $\dfrac{\partial f}{\partial m} = 0$, i.e. when

$$m = \tanh\left(\frac{m+h}{\theta}\right)$$

which is the MF eqn. This may be written as

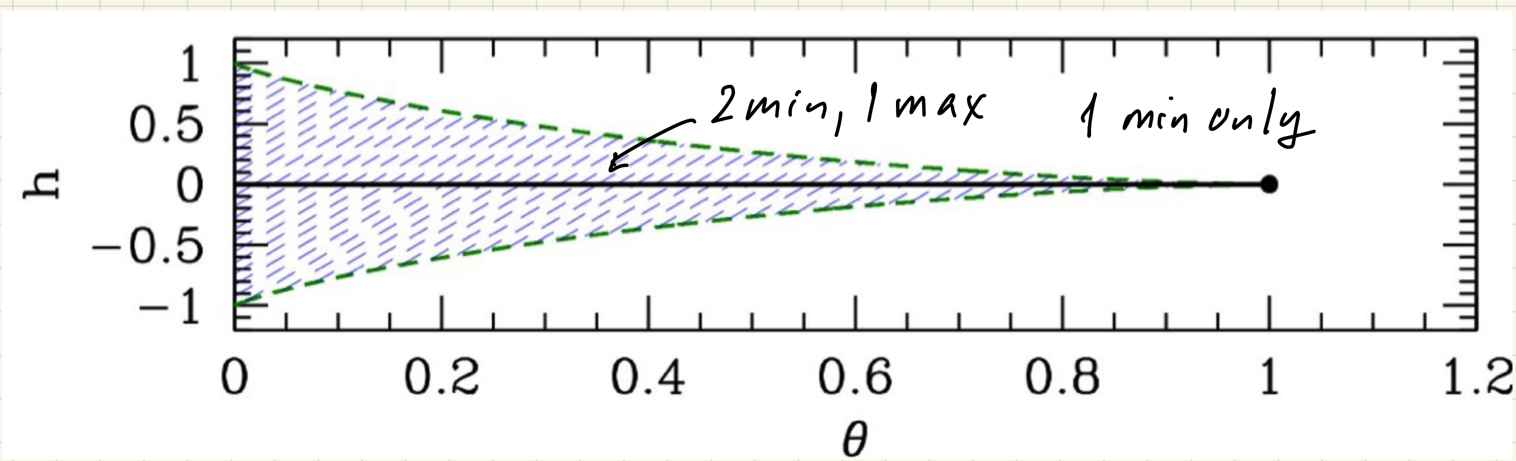
$$h(\theta, m) = \frac{\theta}{2} \log\left(\frac{1+m}{1-m}\right) - m$$



For $\theta < \theta_c = 1$, the free energy has 3 extrema (2 local minima and 1 local maximum) provided $|h| < h^*(\theta)$, where

$$h^*(\theta) = \sqrt{1-\theta} - \frac{\theta}{2}\log\left(\frac{1+\sqrt{1-\theta}}{1-\sqrt{1-\theta}}\right)$$

At $h = \pm h^*(\theta)$ there is a saddle-node bifurcation where a min and max annihilate, leaving 1 min.



When $|h| > h^*(\theta)$ the free energy $f(m,\theta,h)$ has only a single local minimum at $m^*(\theta,h)$. This is a SFP for the dynamics $\dot{m} = -\partial_m f(m,\theta,h)$. When $|h| < h^*(\theta)$ then $f(m,\theta,h)$ has two local minima $m^*_{1,2}(\theta,h)$, where $\mathrm{sgn}(m^*_1) = \mathrm{sgn}(h)$ and $\mathrm{sgn}(m^*_2) = -\mathrm{sgn}(h)$. Both are SFPs of the dynamics but m^*_1 is the global minimum, so m^*_2 corresponds to a metastable state – a "false vacuum". There is also a local maximum at m^*_3, which is a UFP. Slowly ramping the field h up and down then leads to the phenomenon of hysteresis.