Mean Field Theory

Ising model: $\hat{H} = -J\sum_{\langle ij \rangle} \sigma_i \sigma_j - H\sum_i \sigma_i$

where $J_{ij} = J(|\vec{R}_i - \vec{R}_j|)$ is the coupling between sites i and j. We begin by writing $\sigma_i \equiv m_i + \delta\sigma_i$ where $m_i \equiv \langle \sigma_i \rangle$ is a thermodynamic average. Thus $\delta\sigma_i = \sigma_i - m_i$. We assume $m_i = m$ is site independent. Then $\sigma_i \sigma_j = -m^2 + m(\sigma_i + \sigma_j) + \delta\sigma_i \delta\sigma_j$ and we drop the last term which is quadratic in the fluctuations. We then arrive at the mean field Hamiltonian:

$$\hat{H}_{MF} = \frac{1}{2} N z J m^2 - (\underbrace{zJm + H}_{H_{eff}}) \sum_i \sigma_i$$

The effective field is

$$H_{eff} = \underbrace{zJm}_{\text{internal}} + \underbrace{H}_{\text{external}}$$

Thus the problem reduces to noninteracting spins in an external field H_{eff}. We now have

$$Z_{MF} = \mathrm{Tr}\, e^{-\beta \hat{H}_{MF}}$$
$$= e^{-\frac{1}{2}\beta N z J m^2} \left(\sum_{\sigma = \pm} e^{\beta(H + zJm)\sigma} \right)^N$$

Adimensionalize: $f \equiv F/NzJ$, $\theta \equiv k_B T/zJ$, $h \equiv H/zJ$

We then have

$$f(m,\theta,h) = \frac{1}{2}m^2 - \theta \log \cosh\left(\frac{m+h}{\theta}\right) - \theta \log 2$$

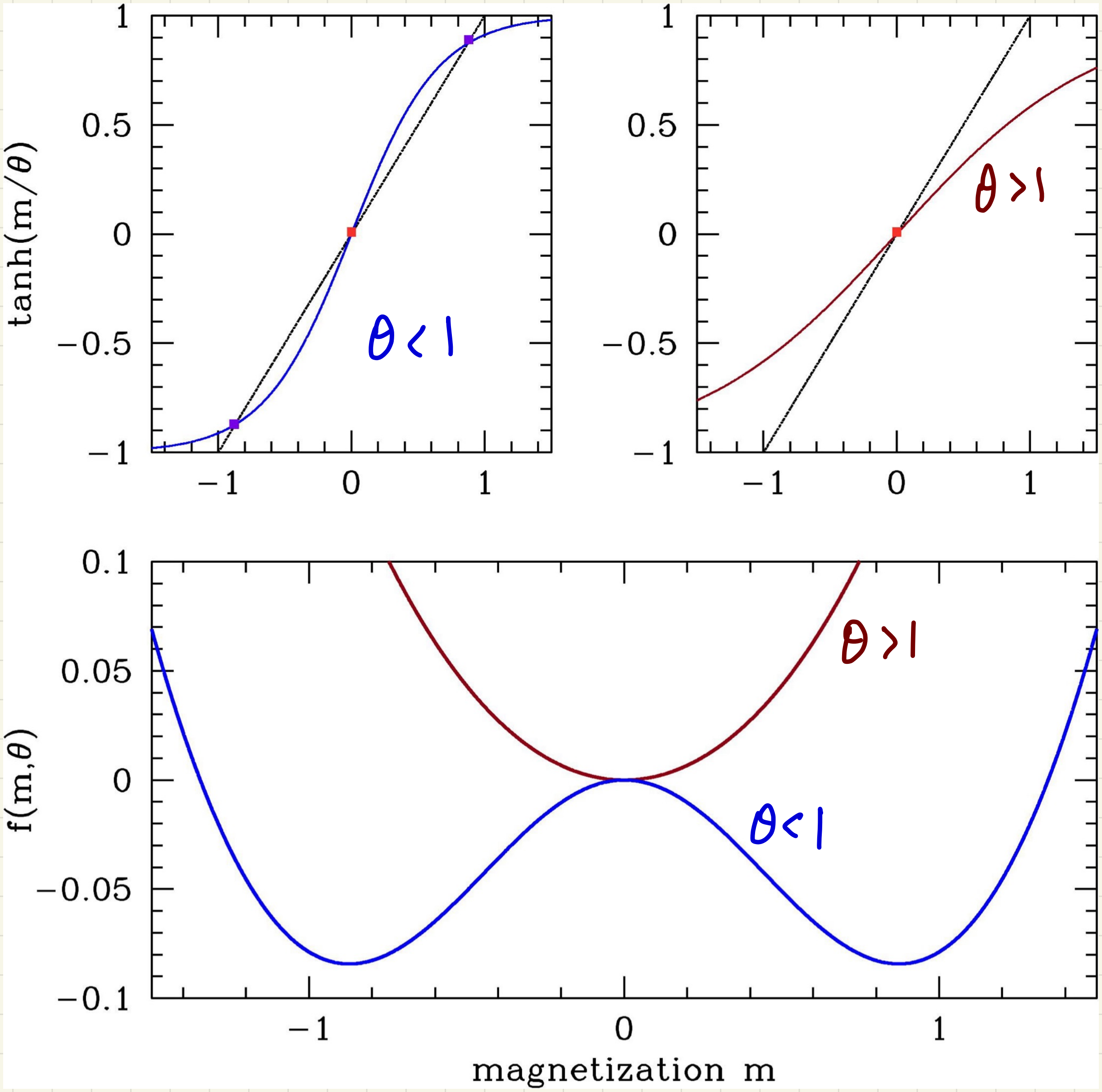
Extremize wrt the mean field m:

$$\frac{\partial f}{\partial m} = 0 \Rightarrow m = \tanh\left(\frac{m+h}{\theta}\right)$$ self-consistent condition

which is $\langle \sigma_i \rangle = \tanh\left(\frac{H_{eff}}{k_B T}\right)$.

<u>Zero external field</u>

When $h = 0$ we have $m = \tanh(m/\theta)$. Let's look at this graphically:



The critical temperature is $\theta_c = 1$. For $\theta < \theta_c$ the broken symmetry ($m \neq 0$) solutions to the self-consistency equation correspond to lower free energy than the symmetry unbroken ($m = 0$) solution (see figure). How does m depend on θ?

Using $\log\cosh u = u^2 - \frac{u^4}{12} + \dots$ we have

$$f(m,\theta,h) = -\theta\log 2 + \frac{1}{2}m^2 - \theta\log\cosh\left(\frac{m}{\theta}\right)$$

$$= -\theta\log 2 + \frac{1}{2}\left(1-\frac{1}{\theta}\right)m^2 + \frac{m^4}{12\theta^3} + \dots$$

$$\frac{\partial f}{\partial m} = 0 = \left(1-\frac{1}{\theta}\right)m + \frac{m^3}{3\theta^3} + \dots$$

Dividing by m, we find

$$m = \pm\sqrt{3}\,(\theta_c-\theta)_+^{1/2}$$ $\leftarrow$ no $m \neq 0$ sol^n for $\theta > 1$