The Ising Model

Ingredients: N-site lattice $\mathcal{L}$, typically with periodic boundary conditions. At each site i lies a spin $\sigma_i = \pm 1$. Now consider

$$\hat{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \mu_0 H \sum_i \sigma_i$$

nearest neighbor link $\rightarrow \langle ij \rangle$

This is the Ising Hamiltonian. We are interested in the $|\mathcal{L}| = N \rightarrow \infty$ (thermodynamic) limit.

Solvable exactly in $d=1$ and in $d=2$ with $H=0$. Else numerical approaches (Monte Carlo), field theory and RG, series expansions.

Correlation f$^{\text{ns}}$: $C_{ij} \equiv \langle \sigma_i \sigma_j \rangle$ ("correlator")

$\tilde{C}_{ij} \equiv \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ ("connected correlator")

When $J=0$, $\langle \sigma_i \rangle = \tanh(\mu_0 H / k_B T)$

$$\tilde{C}_{ij} = 0 \text{ for } i \neq j$$

When $J \neq 0$, different sites are correlated: $\tilde{C}_{ij} \neq 0$

$H=0$: $\mathbb{Z}_2$ symmetry; $E[\vec{\sigma}] = E[\varepsilon \vec{\sigma}]$ where $\vec{\sigma} = \{\sigma_1, \dots, \sigma_N\}$ and $\varepsilon \vec{\sigma} = \{-\sigma_1, -\sigma_2, \dots, -\sigma_N\}$

This entails $\langle \sigma_i \rangle = 0$. At $T=0$ there are two ground states: $|\Uparrow\rangle = |\uparrow\uparrow\uparrow\ldots\uparrow\rangle$

$$|\Downarrow\rangle = |\downarrow\downarrow\downarrow\ldots\downarrow\rangle$$

The energy of each is $E_0 = -N_{links}J$ where $N_{links} = \frac{1}{2}zN$ on a lattice of coordination number z.

The $T=0$ density matrix is $\rho_0 = \frac{1}{2}|\Uparrow\rangle\langle\Uparrow| + \frac{1}{2}|\Downarrow\rangle\langle\Downarrow|$ from which we have $m_i(T,H=0) = \langle\sigma_i\rangle = 0$ and $\tilde{C}_{ij}(T,H=0) = 1$. Note:

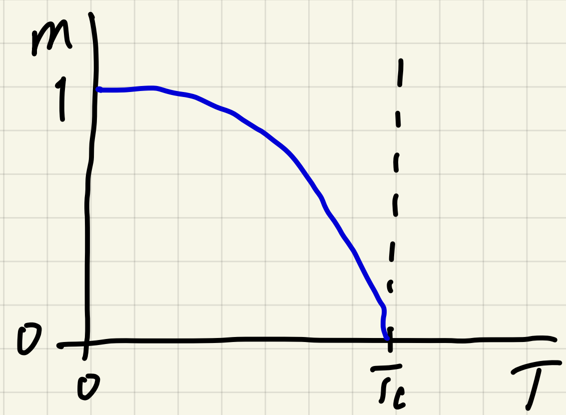
$$\rho(T) = \sum_{\vec{\sigma}} e^{-\beta E(\vec{\sigma})}|\vec{\sigma}\rangle\langle\vec{\sigma}| \Big/ \sum_{\vec{\mu}} e^{-\beta E(\vec{\mu})}$$

Spontaneous symmetry breaking for $H=0$:

① $C_{ij}(T) = \langle\sigma_i\sigma_j\rangle \xrightarrow[\substack{d_{ij}\to\infty \\ H=0}]{} m^2(T)$ with $m^2(T \geq T_c) = 0$

② $m(T) = \lim\limits_{H\to 0}\lim\limits_{N\to\infty}\langle\sigma_i\rangle$ order of limits important!

$m(T<T_c) \neq 0$, $m(T \geq T_c) = 0$



$d=1$ Solution

Choose periodic boundary conditions on a chain of length L. Then

$$\hat{H} = -J\sum_{n=1}^{L}\sigma_n\sigma_{n+1} - \mu_0 H\sum_{n=1}^{L}\sigma_n$$

$$Z(T,L,H) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \mathrm{Tr}\, R^L$$

with $R_{\sigma\sigma'} = e^{\beta J\sigma\sigma'}e^{\beta\mu_0 H(\sigma+\sigma')/2}$

$$= \begin{array}{cc} \begin{pmatrix} e^{\beta(J+\mu_0 H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu_0 H)} \end{pmatrix}_{\sigma\sigma'} & \begin{array}{c} + \\ - \end{array} \\ \begin{array}{cc} + & \quad - \end{array} & \end{array}$$

Note $R = e^{\beta J}\cosh(\beta\mu_0 H)\,\mathbb{1} + e^{-\beta J}X + e^{\beta J}\sinh(\beta\mu_0 H)\,Z$

Then $Z(T,L,H) = \lambda_+^L + \lambda_-^L$ where $\det(\lambda - R) = 0$

$$\Rightarrow \lambda_\pm(T,H) = e^{\beta J}\cosh(\beta\mu_0 H) \pm \sqrt{e^{2\beta J}\sinh^2(\beta\mu_0 H) + e^{-2\beta J}}$$

As $L\to\infty$ the largest eigenvalue dominates and

$$F(T,L,H) = -Lk_B T\log\lambda_+(T,H)$$

Then the magnetization is

$$M = -\frac{\partial F}{\partial H}\bigg|_{T,L} = \frac{L\mu_0 \sinh(\beta\mu_0 H)}{\sqrt{\sinh^2(\beta\mu_0 H) + e^{-4\beta J}}}$$

The magnetic susceptibility is

$$\chi = \frac{1}{L}\frac{\partial M}{\partial H}\bigg|_{H=0} = \frac{\mu_0^2}{k_B T} e^{2J/k_B T}$$

As $J \to 0$ we recover the Curie susceptibility $\chi(T) = \frac{\mu_0^2}{k_B T}$.

Correlation functions:

$$\langle \sigma_1 \sigma_{1+n} \rangle = \mathrm{Tr}(Z R^n Z R^{L-n}) / \mathrm{Tr}\, R^L \equiv C(n)$$

Then with $R = \lambda_+ |+\rangle\langle +| + \lambda_- |-\rangle\langle -|$ we have

$$C(n) = \frac{\lambda_+^L Z_{++} Z_{++} + \lambda_-^L Z_{--} Z_{--} + (\lambda_+^{L-n}\lambda_-^n + \lambda_-^{L-n}\lambda_+^n) Z_{+-} Z_{-+}}{\lambda_+^L + \lambda_-^L}$$

where $Z_{\sigma\sigma'} = \langle \sigma | Z | \sigma' \rangle$.

For $H = 0$ we have $R = e^{\beta J} \mathbb{1} + e^{-\beta J} X$. Then

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \text{ and}$$

$Z_{+-} = Z_{-+} = 1$, $Z_{++} = Z_{--} = 0$. Thus

$$C(n) = \langle \sigma_1 \sigma_{1+n} \rangle = \frac{\lambda_+^{L-|n|}\lambda_-^{|n|} + \lambda_-^{L-|n|}\lambda_+^{|n|}}{\lambda_+^L + \lambda_-^L}$$

$$= \frac{\tanh^{|n|}(\beta J) + \tanh^{L-|n|}(\beta J)}{1 + \tanh^L(\beta J)}$$

$$\approx \tanh^{|n|}(\beta J) \quad \text{for } L \to \infty$$

Writing $C(n) = e^{-|n|/\xi(T)}$ we have

$$\xi(T) = \frac{1}{\log \operatorname{ctnh}(J/k_B T)}$$

and as $T \to 0$ we obtain $\xi(T \to 0) \approx \frac{1}{2} e^{2J/k_B T}$

Domain walls: $\uparrow\uparrow\uparrow\uparrow | \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow | \uparrow\uparrow\uparrow\uparrow$

DW energy is $E(\uparrow\downarrow) - E(\uparrow\uparrow) = 2J$. Thus

$$F = 2MJ - k_B T \log\binom{L}{M} + E_0$$

$E_0 = -LJ$

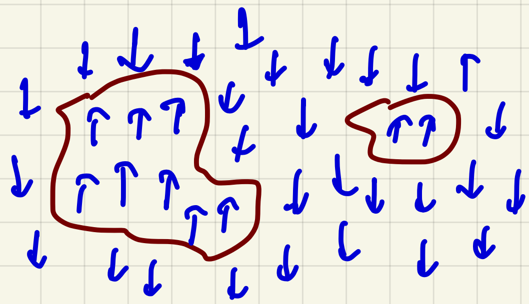
$$= L \cdot \left\{ 2Jx + k_B T \left[x \log x + (1-x)\log(1-x) \right] \right\} + E_0$$

where $M = \#$ DW and $x \equiv \frac{M}{L}$ = DW density

Setting $\frac{\partial F}{\partial x} = 0 \Rightarrow x = \frac{1}{e^{2J/k_B T}+1}$

This is finite for all $T > 0$, which is consistent with no LR spin order.

Higher dimensions:



$$Z = 2\sum_{\Gamma} e^{-2\beta J L_\Gamma}$$

where Γ is a configuration of loops. Thus

$$Z = 2\sum_{P=0}^{\infty} N_P e^{-2\beta J P} \times e^{-\beta E_0}$$

P = total perimeter

and $E_0 = -N_{links} J = -\frac{1}{2} N z J$.

Approximation: $N_P \approx (z-1)^P \Rightarrow$

$$N_P e^{-2\beta J P} \approx \exp\left\{\left[\log(z-1) - 2\beta J\right] P\right\}$$

and DWs proliferate for

$$k_B T > k_B T_c \approx \frac{2J}{\log(z-1)} = 1.82J$$

(exact $2.27J$)