Dist^{n} of blackbody radiation:

EM radiation: $\nu = \frac{c}{\lambda} = \frac{ck}{2\pi}$, $\varepsilon_k = \hbar c k$

photons per unit frequency at temperature T :

$$N(\nu, T)\, d\nu = \frac{2V}{(2\pi)^3} \frac{d^3k}{e^{\hbar c k / k_B T} - 1}$$

Energy:

$$\mathcal{E}(\nu, T) = \frac{8\pi \hbar V}{c^3} \frac{\nu^3}{e^{h\nu/k_B T} - 1}$$

$$= N(\nu, T) \times h\nu \quad ; \quad k = \frac{2\pi\nu}{c} \text{ etc.}$$

Total energy:

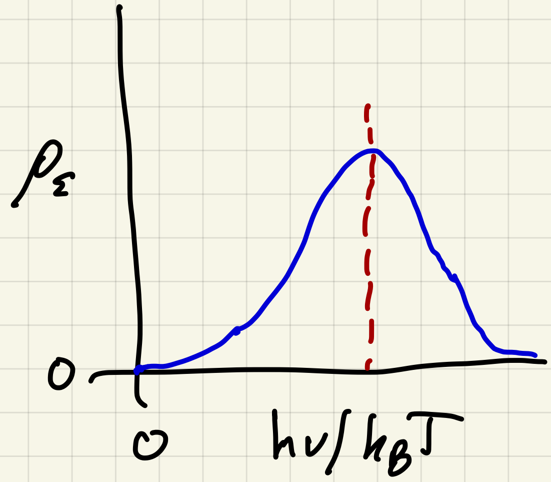
$$E(T) = \int_0^\infty d\nu\, \mathcal{E}(\nu, T) = V \cdot \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3}$$

Spectral density $\rho_\varepsilon(\nu, T)$:

$$\rho_\varepsilon(\nu, T) = \frac{\mathcal{E}(\nu, T)}{E(T)} = \frac{15}{\pi^4} \frac{h}{k_B T} \cdot \frac{(h\nu/k_B T)^3}{e^{h\nu/k_B T} - 1}$$

Scaled frequency $s \equiv h\nu / k_B T$ (dim$^{\text{less}}$) $\Rightarrow$

$$\rho_\varepsilon = \text{const} \cdot \frac{s^3}{e^s - 1}$$



$$\frac{d}{ds}\left(\frac{s^3}{e^s - 1}\right) = 0$$

$$\Rightarrow \quad s^* = 2.82144$$

$$\nu_{max} = s^* \cdot \frac{k_B T}{h} \propto T$$

Photons γ $(d=3)$

- dispersion $\varepsilon_{\vec{k}} = \hbar c|\vec{k}|$, $\vec{k} \in \mathbb{R}^3$

$$\hat{H} = \sum_{\vec{k},\sigma} \varepsilon_{\vec{k}} \hat{n}_{\vec{k}\sigma} \quad ; \quad n_{\vec{k}\sigma} \in \{0,1,\ldots\}$$

polarization

- bosons with $\mu = 0$, $g_d = 2$ (polarization)

- $n_{\vec{k}} = \langle \hat{n}_{\vec{k}} \rangle = \dfrac{1}{e^{\beta \varepsilon_{\vec{k}}} - 1}$ occupancy of $|\vec{k},\sigma\rangle$

- DOS $g(\varepsilon) = \dfrac{\varepsilon^2}{\pi^2 (\hbar c)^3}$ $(\varepsilon \geq 0)$

- $n(T) = \displaystyle\int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{e^{\beta\varepsilon} - 1} = \frac{2\zeta(3)}{\pi^2} \frac{(k_B T)^3}{(\hbar c)^3}$

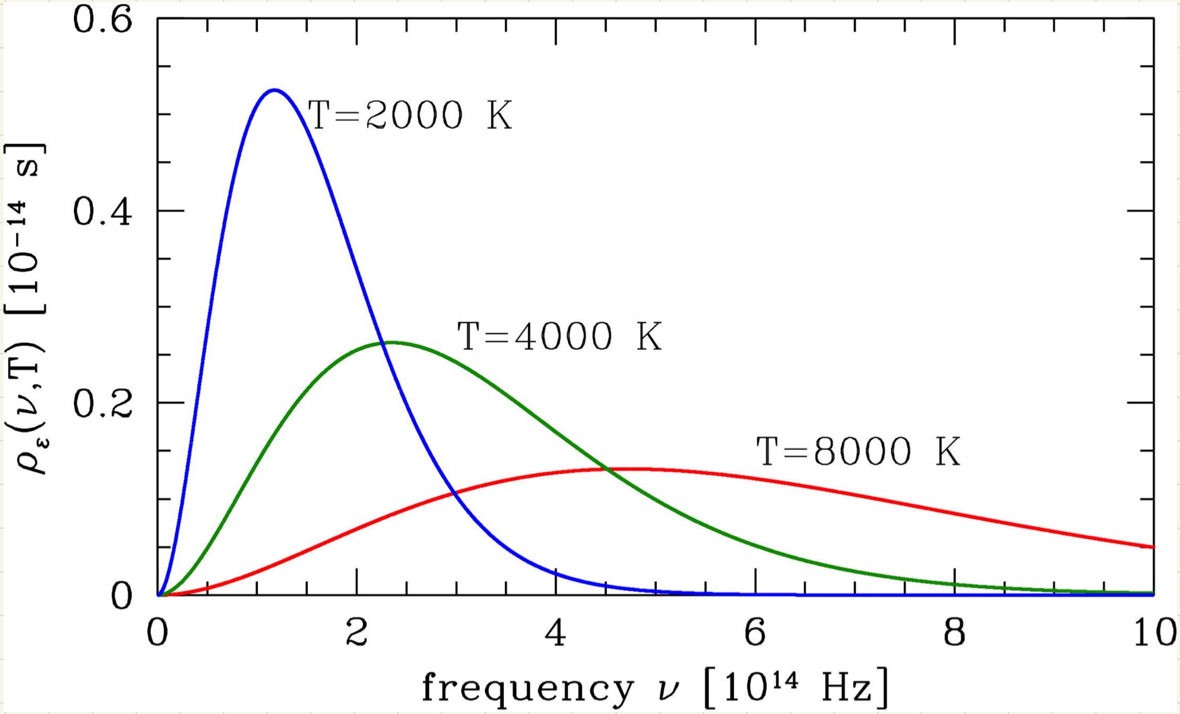
- $p(T) = -k_B T \displaystyle\int_0^\infty d\varepsilon \, g(\varepsilon) \log\left(1 - e^{-\beta\varepsilon}\right)$

$$= \frac{2\zeta(4)}{\pi^2} \frac{(k_B T)^4}{(\hbar c)^3}$$

- blackbody spectral density $\rho_\varepsilon(\nu, T)$

$$\rho_\varepsilon(\nu,T) = \frac{15}{\pi^4}\frac{h}{k_B T}\frac{(h\nu/k_B T)^3}{e^{h\nu/k_B T}-1}$$

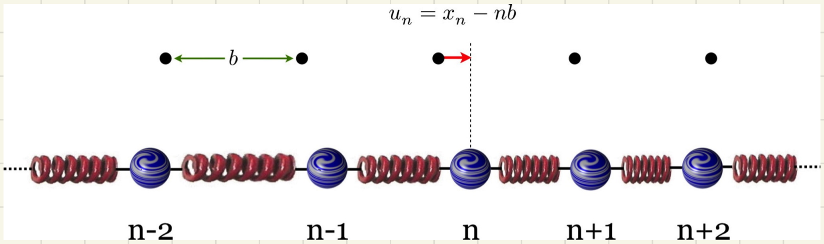
peaked at $\nu = 2.82\, k_B T/h$



$\int_{\nu_1}^{\nu_2} d\nu\, \rho_\varepsilon(\nu,T)$ = fraction of $E(T)$ in modes with $\nu \in [\nu_1, \nu_2]$

Phonons

- lattice vibrations in solids



Ideal Bose Gas

Classical NRIG: $Z = \frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N \Rightarrow$

$$F(T,V,N) = -k_B T \log Z(T,V,N)$$

$$= -Nk_B T \log\left(\frac{V}{N\lambda_T^3}\right) - Nk_B T$$

Use: $\log N! \simeq N\log N - N + \mathcal{O}(\log N)$

Entropy:

$$S = -\frac{\partial F}{\partial T}\bigg|_{V,N} = Nk_B \log\left(\frac{V}{N\lambda_T^3}\right) + \frac{5}{2}Nk_B$$

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mk_B T}\right)^{1/2} \Rightarrow S \sim Nk_B \times \frac{3}{2}\log T$$

$$S(T\to 0^+) = -\infty$$

$$\underset{q}{S}(T=0) = k_B \log g_d$$

Noninteracting bosons: $\hat{h}|\alpha\rangle = \varepsilon_\alpha |\alpha\rangle$

1-body Hamiltonian, 1-body spectrum

Occupancies:

$$n_\alpha = \langle \hat{n}_\alpha \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} - 1}$$

Number density: $n_0 = \dfrac{1}{z^{-1} - 1}$

$$n(T,\mu) = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu}\Big|_{T,V}$$

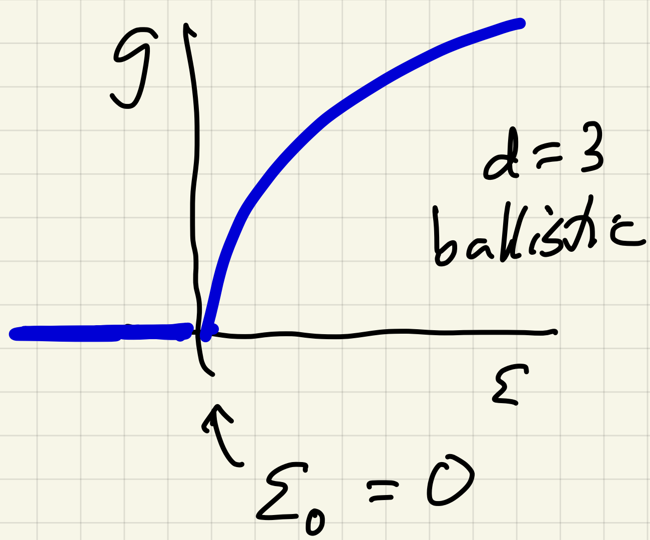
$$= \frac{1}{V} \sum_\alpha n_\alpha = \int d\varepsilon \, \frac{g(\varepsilon)}{e^{\beta(\varepsilon - \mu)} - 1}$$

$$g(\varepsilon) = \frac{1}{V} \sum_\alpha \delta(\varepsilon - \varepsilon_\alpha) \propto \varepsilon^{\frac{d}{2} - 1}$$

E.g. particles with $S = \frac{1}{2}$

$$\varepsilon(\vec{k}, \sigma) = \frac{\hbar^2 k^2}{2m} - \mu_0 B \sigma$$

$$\Rightarrow \varepsilon_0 = -\mu_0 B$$



$$n(T,\mu) = \int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{e^{(\varepsilon-\mu)/k_BT} - 1}$$

$\varepsilon_0 = 0$

$$= \int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{z^{-1} e^{\varepsilon/k_BT} - 1} = n(T,z)$$

Since $\mu < \varepsilon_0 = 0 \Rightarrow z = e^{\beta\mu} \in [0,1)$

$\mu \in (-\infty, 0)$ $\qquad \mu = -\infty \qquad \mu \to 0^-$

Clearly $n(T,z)$ increasing in z at fixed T

I.e. $\frac{\partial n}{\partial z} > 0$. What is maximum allowed n?

$$n_{max} = n(T, z=1) = \int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{e^{\varepsilon/k_BT} - 1}$$

Or, given n, there is a critical temperature $T_c(n)$ such that

$$n = \int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{e^{\varepsilon/k_B T_c} - 1}$$

Implicit eqn for $T_c(n)$.

Need to separate out $\vec{k}=0$ mode:

$$n(T,\mu) = \underbrace{\frac{1}{V}\,\frac{1}{z^{-1}-1}}_{n_0(T)} + \underbrace{\int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{z^{-1}e^{\varepsilon/k_B T}-1}}_{n'(T)}$$

$n_c(T)$: condensate number density

$n'(T)$ = overcondensate

If n_0 finite, then

$$V n_0 = \frac{1}{z^{-1}-1} \Rightarrow z^{-1} = 1 + \frac{1}{V n_0}$$

$$= e^{-\mu/k_B T}$$