

**PHYSICS 140B W26 : STATISTICAL PHYSICS  
HW ASSIGNMENT #7 SOLUTIONS**

**(1)** Consider the collisionless Boltzmann equation for the Hamiltonian  $\hat{H}(p) = \frac{1}{4}Ap^4$  in one space dimension. Suppose the initial distribution is given by

$$f(x, p, t = 0) = C e^{-x^2/2\sigma^2} e^{-p^2/2\kappa^2} .$$

(a) Find  $f(x, p, t)$  for all  $t > 0$ .

(b) Find the equation for the locus of points  $(x, p)$  for which  $f(x, p, t) = C \exp(-\alpha^2/2)$ .

(c) Express your result in (b) in dimensionless form and plot it for various values of the dimensionless time.

**Solution :**

(a) The velocity is  $v(p) = \partial\varepsilon/\partial p = Ap^3$  and thus

$$f(x, p, t) = f(x - v(p)t, p, 0) = C e^{-(x - Ap^3t)^2/2\sigma^2} e^{-p^2/2\kappa^2} .$$

(b) Clearly we have

$$\frac{(x - Ap^3t)^2}{\sigma^2} + \frac{p^2}{\kappa^2} = \alpha^2 ,$$

which may be written as

$$x(p, t; \alpha) = Ap^3t \pm \sigma \sqrt{\alpha^2 - \frac{p^2}{\kappa^2}} .$$

If we write

$$\bar{x} \equiv \frac{x}{\sigma} , \quad \bar{p} \equiv \frac{p}{\kappa} , \quad s \equiv \frac{A\kappa^3 t}{\sigma} ,$$

then in dimensionless form we have

$$\bar{x}(\bar{p}, s; \alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2} .$$

(c) See fig. 1.

**(2)** Consider an ideal gas of point particles in  $d = 3$  dimensions with isotropic dispersion  $\varepsilon(\mathbf{p}) = A|\mathbf{p}|^4$ .

(a) Find the enthalpy per particle  $h = \mu + Ts$ , where  $\mu$  is the chemical potential and  $s$  is the entropy per particle. (You may find it useful to review some of the material in chapter 4 of the notes.)

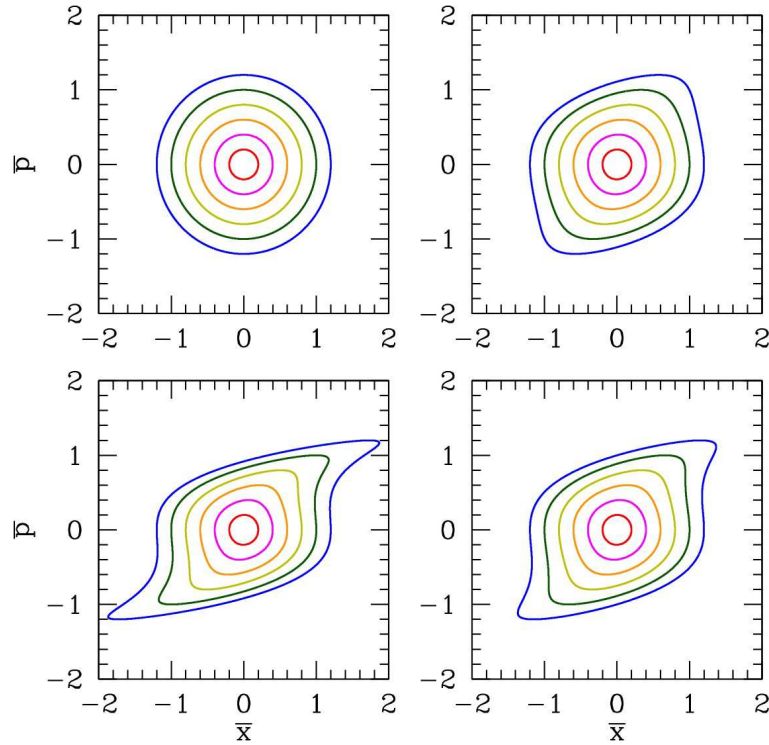


Figure 1: Level sets  $\bar{x}(\bar{p}, s; \alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2}$  for  $\alpha = 0.2$  (red) to  $\alpha = 1.2$  (blue) and  $s = 0$  (upper left),  $s = \frac{1}{3}$  (upper right),  $s = \frac{2}{3}$  (lower right), and  $s = 1$  (lower left). Compare with fig. 8.1 of the lecture notes.

(b) Find the thermal conductivity  $\kappa$  within the relaxation time approximation.

**Solution :**

(a) Recall that for the dispersion  $\varepsilon(p) = Ap^\sigma$ , in  $d$  dimensions, the density of states per unit volume is given by  $g(\varepsilon) = C\varepsilon^{r-1}$  where  $r = d/\sigma$  and  $C$  is a constant. One then obtains the Helmholtz free energy

$$\begin{aligned}
 F &= -Nk_B T \log(V/N) - Nk_B T - Nk_B T \log \int_0^\infty d\varepsilon g(\varepsilon) \exp(-\varepsilon/k_B T) \\
 &= -Nk_B T \log(V/N) - Nk_B T \log[eA\Gamma(r)] - rNk_B T \log(k_B T) \quad .
 \end{aligned}$$

We then have

$$\begin{aligned}
 \mu &= \left( \frac{\partial F}{\partial N} \right)_{T,V} = -k_B T \log(V/N) - k_B T \log[A\Gamma(r)] - rk_B T \log(k_B T) \\
 s &= -\frac{1}{N} \left( \frac{\partial F}{\partial T} \right)_{V,N} = k_B \log(V/N) + k_B \log[eA\Gamma(r)] + rk_B \log(ek_B T) \quad ,
 \end{aligned}$$

and thus

$$h = \mu + Ts = (r + 1)k_B \quad .$$

The thermal conductivity is obtained from the energy current  $\mathbf{j}_\varepsilon = -\kappa \nabla T$ , where

$$j_\varepsilon^\alpha = \int d^d p \varepsilon v^\alpha \delta f = -\frac{n\tau}{k_B T^2} \langle v^\alpha v^\beta \varepsilon (\varepsilon - h) \rangle \frac{\partial T}{\partial x^\beta} ,$$

where we have used the result (see eqn. 8.71 of the lecture notes)

$$\delta f = -\frac{\tau}{k_B T^2} (\varepsilon - h) (\mathbf{v} \cdot \nabla T) f^0$$

and  $h = c_p T$  for our ideal gas. Thus from isotropy we can replace  $\langle v^\alpha v^\beta \rangle \rightarrow d^{-1} \langle v^2 \rangle \delta^{\alpha\beta}$ , in which case

$$\kappa = \frac{n\tau}{dk_B T^2} \langle \mathbf{v}^2 \varepsilon (\varepsilon - h) \rangle .$$

With  $\varepsilon(p) = Ap^\sigma$  we have  $\mathbf{v}(p) = \sigma Ap^{\sigma-1} \hat{\mathbf{p}}$  and thus  $\mathbf{v}^2 = \sigma^2 A^{2/\sigma} \varepsilon^{2-2\sigma^{-1}}$ . We then have

$$\kappa = \frac{n\tau}{dk_B T^2} \sigma^2 A^{2/\sigma} \langle \varepsilon^{3-2\sigma^{-1}} (\varepsilon - (r+1)k_B T) \rangle$$

Now with the density of states  $g(\varepsilon) \propto \varepsilon^{r-1}$  we have

$$\langle \varepsilon^s \rangle = \int_0^\infty d\varepsilon \varepsilon^{r-1+s} e^{-\beta\varepsilon} / \int_0^\infty d\varepsilon \varepsilon^{r-1} e^{-\beta\varepsilon} = \frac{\Gamma(r+s)}{\Gamma(r)} (k_B T)^s .$$

Working out the remaining details, we arrive at the expression

$$\kappa = \frac{2\sigma(\sigma-1)\Gamma(r+3-2\sigma^{-1})}{d\Gamma(r)} A^{2/\sigma} \cdot n\tau k_B \cdot (k_B T)^{2(1-\sigma^{-1})} .$$

As a check, if we set  $\sigma = 2$  and  $A = 1/2m$  (ballistic dispersion), then for  $d = 3$  we have  $r = \frac{3}{2}$  and find  $\kappa = \frac{5}{2} n\tau k_B^2 T/m$ , which is identical to the result in §8.5.3 of the lecture notes. Setting  $\sigma = 4$  and  $d = 3$ , we obtain

$$\kappa = \frac{8\Gamma(\frac{13}{4})}{\Gamma(\frac{3}{4})} A^{1/2} n\tau k_B (k_B T)^{3/2} = 16.6425 A^{1/2} n\tau k_B (k_B T)^{3/2} .$$

**(3)** Consider a nonequilibrium distribution of the form

$$f(\mathbf{r}, \mathbf{p}, t=0) = h^{-3} n \lambda_T^3 e^{-\mathbf{p}^2/2mk_B T} \left( 1 + \frac{\alpha \mathbf{p}^2}{2mk_B T} \right)$$

and investigate its relaxation to the equilibrium distribution  $f^0(\mathbf{p}) = h^{-3} n \lambda_T^3 e^{-\mathbf{p}^2/2mk_B T}$  using the Boltzmann equation in the relaxation time approximation, with no external forces. Find  $f(\mathbf{r}, \mathbf{p}, t)$ . Then find  $N(t)$  and  $E(t)$ , the time-dependent values for the total particle number and total energy. You may abbreviate  $N_0 \equiv nV$ , where  $V$  is the system

volume and  $N_0$  is the number of particles at equilibrium. Then, drawing upon your understanding of collisional invariants, explain why your calculation is complete BS. What has gone wrong?

**Solution :**

It is convenient to express everything in terms of the single particle energy  $\varepsilon = \mathbf{p}^2/2m$ :

$$f(\mathbf{p}, t = 0) = n\lambda_T^3 e^{-\beta\varepsilon} (1 + \alpha \cdot \beta\varepsilon) = f^0 (1 + \alpha \cdot \beta\varepsilon) \quad ,$$

where  $\beta = 1/k_B T$  as usual. In the absence of temperature or velocity gradients, and with no external forces, the Boltzmann equation takes the form

$$\frac{\partial \delta f}{\partial t} = \left( \frac{df}{dt} \right)_{\text{coll}} \quad .$$

Within the relaxation time approximation, then,

$$\frac{\partial \delta f}{\partial t} = -\frac{\delta f}{\tau} \quad \Rightarrow \quad \delta f(\mathbf{p}, t) = \delta f(\mathbf{p}, 0) e^{-t/\tau} \quad .$$

So we have

$$f(\mathbf{p}, t) = f^0 + \alpha \beta \varepsilon f^0 e^{-t/\tau} \quad .$$

Recall further that we may write

$$\frac{d^3p}{h^3} f^0(\mathbf{p}) = \frac{2}{\sqrt{\pi}} (k_B T)^{-3} d\varepsilon e^{1/2} e^{-\beta\varepsilon} \equiv n P_{\text{MB}}(\varepsilon) d\varepsilon \quad ,$$

with

$$\langle \varepsilon^s \rangle = \int_0^\infty d\varepsilon P_{\text{MB}}(\varepsilon) \varepsilon^s = \frac{\Gamma(\frac{3}{2} + s)}{\Gamma(\frac{3}{2})} (k_B T)^s \quad .$$

Thus, we find

$$\begin{aligned} N(t) &= N_0 \left( 1 + \frac{3}{2} \alpha e^{-t/\tau} \right) \\ E(t) &= \frac{3}{2} N_0 k_B T \left( 1 + \frac{15}{4} \alpha e^{-t/\tau} \right) \quad , \end{aligned}$$

where  $N_0 = nV$ . We see that neither particle number nor total energy are conserved, which is clearly wrong. This is a defect of the relaxation time approximation. In general, the collision integral must annihilate the five *collisional invariants*, which are total particle number, total energy, and the three components of the total momentum. These quantities are all conserved by the collisions. Every quantity other than these should relax on some microscopic time scale, but the collisional invariants don't relax at all, and the relaxation time approximation (RTA) is too crude to distinguish between the collisional invariants and other quantities – everything relaxes at the same rate within the RTA.

Incidentally, there is a minor modification to the relaxation time approximation collision integral which results in number conservation, although it does not yield energy conservation. This is the so-called BGK collision integral, named for P. L. Bhatnagar, E. P. Gross,

and M. Krook, *Phys. Rev.* **94**, 511 (1954). BGK took

$$\left(\frac{df}{dt}\right)_{\text{coll}} = -\frac{f(\mathbf{r}, \mathbf{p}, t)}{\tau} + \frac{n(\mathbf{r}, t)}{n_0(\mathbf{r})} \frac{f^0(\mathbf{r}, \mathbf{p})}{\tau} ,$$

where

$$n(\mathbf{r}, t) = \int \frac{d^3p}{h^3} f(\mathbf{r}, \mathbf{p}, t) \quad (1)$$

$$n_0(\mathbf{r}) = \int \frac{d^3p}{h^3} f^0(\mathbf{r}, \mathbf{p}) . \quad (2)$$

Note then that

$$\int \frac{d^3p}{h^3} \left(\frac{df}{dt}\right)_{\text{coll}} = 0 ,$$

and there is local number conservation by collisions<sup>1</sup>. The nice thing about the BTG collision integral is that it is still linear in  $f$ .

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<sup>1</sup>The BGK collision integral also can be used in the presence of streaming terms.