PHYSICS 140B W26 : STATISTICAL PHYSICS
HW ASSIGNMENT #3 SOLUTIONS

(1) For ideal Fermi gases in $d = 1$, 2, and 3 dimensions, compute at $T = 0$ the average fermion speed.

Solution :

At $T = 0$ the average speed $v = |\boldsymbol{v}|$ is

$$\langle v \rangle = \int_0^{k_F} dk\, k^{d-1}\, \frac{\hbar k}{m} \Bigg/ \int_0^{k_F} dk\, k^{d-1} = \frac{d}{d+1} \cdot \frac{\hbar k_F}{m} \ .$$

The number density is

$$n = \frac{\mathsf{g}\,\Omega_d}{(2\pi)^d} \int_0^{k_F} dk\, k^{d-1} = \frac{\mathsf{g}\,\Omega_d\, k_F^d}{(2\pi)^d\, d} \qquad \Rightarrow \qquad k_F = 2\pi \bigg(\frac{d}{\mathsf{g}\,\Omega_d}\bigg)^{1/d} n^{1/d} \ .$$

Putting these together we can obtain the average velocity in terms of the density n and physical constants.

(2) Consider a two-dimensional gas of fermions which obey the dispersion relation

$$\varepsilon(\boldsymbol{k}) = \varepsilon_0 \Big((k_x^2 + k_y^2)\, a^2 + \tfrac{1}{2}(k_x^4 + k_y^4)\, a^4 \Big) \ .$$

Sketch, on the same plot, the Fermi surfaces for $\varepsilon_F = 0.1\,\varepsilon_0$, $\varepsilon_F = \varepsilon_0$, and $\varepsilon_F = 10\,\varepsilon_0$.

Solution : It is convenient to adimensionalize, writing

$$x \equiv k_x a \quad , \quad y \equiv k_y a \quad , \quad \nu \equiv \frac{\varepsilon}{\varepsilon_0} \ . \tag{1}$$

Then the equation for the Fermi surface becomes

$$x^2 + y^2 + \tfrac{1}{2}x^4 + \tfrac{1}{2}y^4 = \nu \ . \tag{2}$$

In other words, we are interested in the *level sets* of the function $\nu(x,y) \equiv x^2+y^2+\frac{1}{2}x^4+\frac{1}{2}y^4$. When ν is small, we can ignore the quartic terms, and we have an isotropic dispersion, with $\nu = x^2 + y^2$. *I.e.* we can write $x = \nu^{1/2}\cos\theta$ and $y = \nu^{1/2}\sin\theta$. The quartic terms give a contribution of order ν^4, which is vanishingly small compared with the quadratic term in the $\nu \to 0$ limit. When $\nu \sim \mathcal{O}(1)$, the quadratic and quartic terms in the dispersion are of the same order of magnitude, and the continuous $\mathsf{O}(2)$ symmetry, namely the symmetry under rotation by any angle, is replaced by a discrete symmetry group, which is the group of the square, known as C_{4v} in group theory parlance. This group has eight elements:

$$\big\{ \mathbb{I}\ ,\ R\ ,\ R^2\ ,\ R^3\ ,\ \sigma\ ,\ \sigma R\ ,\ \sigma R^2\ ,\ \sigma R^3 \big\} \tag{3}$$

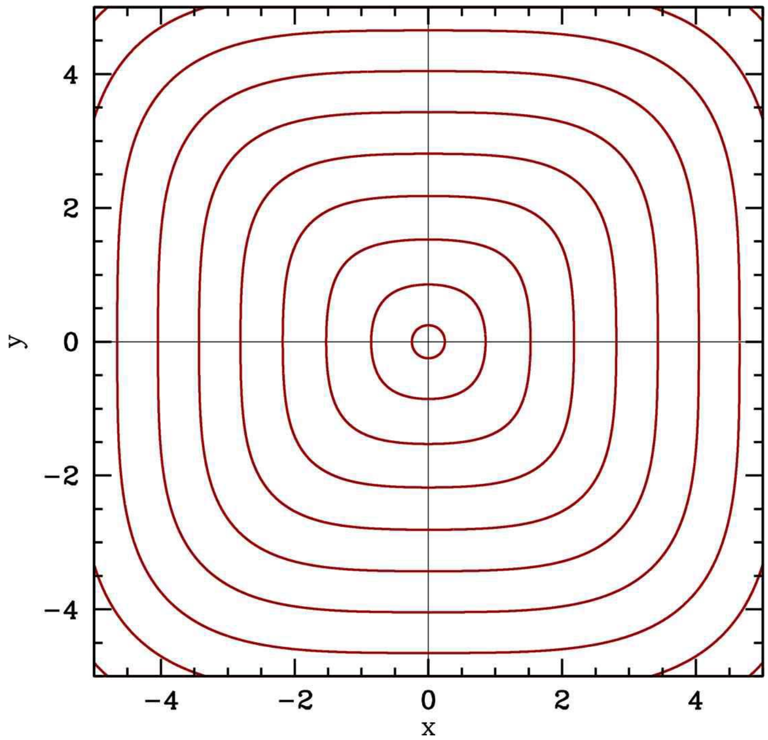

Figure 1: Level sets of the function $\nu(x,y) = x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$ for $\nu = (\frac{1}{2}n)^4$, with positive integer n.

Here R is the operation of counterclockwise rotation by $90°$, sending $(x\,,\,y)$ to $(-y\,,\,x)$, and σ is reflection in the y-axis, which sends $(x\,,\,y)$ to $(-x\,,\,y)$. One can check that the function $\nu(x,y)$ is invariant under any of these eight operations from C_{4v}.

Explicitly, we can set $y = 0$ and solve the resulting quadratic equation in x^2 to obtain the maximum value of x, which we call $a(\nu)$. One finds

$$\tfrac{1}{2}x^4 + x^2 - \nu = 0 \qquad \Longrightarrow \qquad a = \sqrt{\sqrt{1+2\nu} - 1}\ . \tag{4}$$

So long as $x \in \{-a, a\}$, we can solve for $y(x)$:

$$y(x) = \pm\sqrt{\sqrt{1 + 2\nu - 2x^2 - x^4} - 1}\ . \tag{5}$$

A sketch of the level sets, showing the evolution from an isotropic (*i.e.* circular) Fermi surface at small ν, to surfaces with discrete symmetries, is shown in fig. 1.

(3) Obtain numerical estimates for the Fermi energy (in eV) and the Fermi temperature (in Kelvins) for the following systems:

(a) conduction electrons in silver, lead, and aluminum

(b) nucleons in a heavy nucleus, such as ${}^{200}\mathrm{Hg}$

Solution :

(a) The Fermi energy for ballistic dispersion is given by

$$\varepsilon_{\mathrm{F}} = \frac{\hbar^2}{2m^*}\,(3\pi^2 n)^{2/3} \ ,$$

where m^* is the effective mass, which one can assume is the electron mass $m = 9.11 \times 10^{-28}\,\mathrm{g}$. The electron density is given by the number of valence electrons of the atom divided by the volume of the unit cell. A typical unit cell volume is on the order of $30\,\mathrm{\AA}^3$, and if we assume one valence electron per atom we obtain a Fermi energy of $\varepsilon_{\mathrm{F}} = 3.8\,\mathrm{eV}$, and hence a Fermi temperature of $3.8\,\mathrm{eV}/(86.2 \times 10^{-6}\,\mathrm{eV/K}) = 4.4 \times 10^4\,\mathrm{K}$. This sets the overall scale. For detailed numbers, one can examine table 2.1 in *Solid State Physics* by Ashcroft and Mermin. One finds

$$T_{\mathrm{F}}(\mathrm{Ag}) = 6.38 \times 10^4\,\mathrm{K} \quad ; \quad T_{\mathrm{F}}(\mathrm{Pb}) = 11.0 \times 10^4\,\mathrm{K} \quad ; \quad T_{\mathrm{F}}(\mathrm{Al}) = 13.6 \times 10^4\,\mathrm{K} \quad .$$

(b) Nuclear densities are of course much higher. In the literature one finds the relation $R \sim A^{1/3}\,r_0$, where R is the nuclear radius, A is the number of nucleons (*i.e.* the atomic mass number), and $r_0 \simeq 1.2\,\mathrm{fm} = 1.2 \times 10^{-15}\,\mathrm{m}$ Under these conditions, the nuclear density is on the order of $n \sim 3A/4\pi R^3 = 3/4\pi r_0^3 = 1.4 \times 10^{44}\,\mathrm{m}^{-3}$. With the mass of the proton $m_{\mathrm{p}} = 938\,\mathrm{MeV}/c^2$ we find $\varepsilon_{\mathrm{F}} \sim 30\,\mathrm{MeV}$ for the nucleus, corresponding to a temperature of roughly $T_{\mathrm{F}} \sim 3.5 \times 10^{11}\,\mathrm{K}$.

(4) Consider a three-dimensional Fermi gas of $S = \frac{1}{2}$ particles obeying the dispersion relation $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^4$.

(a) Compute the density of states $g(\varepsilon)$.

(b) Compute the molar heat capacity.

(c) Compute the lowest order nontrivial temperature dependence for $\mu(T)$ at low temperatures. *I.e.* compute the $\mathcal{O}(T^2)$ term in $\mu(T)$.

Solution :

(a) The density of states in $d = 3$ ($\mathrm{g} = 2S + 1 = 2$) is given by

$$g(\varepsilon) = \frac{1}{\pi^2}\int\limits_0^\infty \! dk\, k^2\, \delta\big(\varepsilon - \varepsilon(k)\big) = \frac{1}{\pi^2}\, k^2(\varepsilon)\, \frac{dk}{d\varepsilon}\bigg|_{k=(\varepsilon/A)^{1/4}} = \frac{\varepsilon^{-1/4}}{4\pi^2 A^{3/4}} \ .$$

(b) The molar heat capacity is

$$c_V = \frac{\pi^2}{3n}\, R\, g(\varepsilon_{\rm F})\, k_{\rm B}T = \frac{\pi^2 R}{4} \cdot \frac{k_{\rm B}T}{\varepsilon_{\rm F}} \ ,$$

where $\varepsilon_{\rm F} = \hbar^2 k_{\rm F}^2/2m$ can be expressed in terms of the density using $k_{\rm F} = (3\pi^2 n)^{1/3}$, which is valid for any isotropic dispersion in $d = 3$. In deriving this formula we had to express the density n, which enters in the denominator in the above expression, in terms of $\varepsilon_{\rm F}$. But this is easy:

$$n = \int\limits_0^{\varepsilon_{\rm F}} d\varepsilon\, g(\varepsilon) = \frac{1}{3\pi^2} \left(\frac{\varepsilon_{\rm F}}{A}\right)^{3/4} .$$

(c) We have (Lecture Notes, §5.6.4)

$$\delta\mu = -\frac{\pi^2}{6}\,(k_{\rm B}T)^2\, \frac{g'(\varepsilon_{\rm F})}{g(\varepsilon_{\rm F})} = \frac{\pi^2}{24} \cdot \frac{(k_{\rm B}T)^2}{\varepsilon_{\rm F}} \ .$$

Thus,

$$\mu(n,T) = \varepsilon_{\rm F}(n) + \frac{\pi^2}{24} \cdot \frac{(k_{\rm B}T)^2}{\varepsilon_{\rm F}(n)} + \mathcal{O}(T^4) \ ,$$

where $\varepsilon_{\rm F}(n) = \frac{\hbar^2}{2m}\,(3\pi^2 n)^{2/3}$.