PHYSICS 140B W26 : STATISTICAL PHYSICS
HW ASSIGNMENT #2 SOLUTIONS

(1) For a system of noninteracting $S = 0$ bosons obeying the dispersion $\varepsilon(\boldsymbol{k}) = \hbar v|\boldsymbol{k}|$.

(a) Find the density of states per unit volume $g(\varepsilon)$ in d space dimensions.

(b) Determine the critical temperature for Bose-Einstein condensation in $d = 3$ dimensions.

(c) Find the condensate fraction n_0/n for $T < T_c$.

(d) For this dispersion, is there a finite transition temperature in $d = 2$ dimensions? If not, explain why. If so, compute $T_c^{(d=2)}$.

Solution :

(a) The density of states in d dimensions is

$$g(\varepsilon) = \int \frac{d^d k}{(2\pi)^d}\, \delta(\varepsilon - \hbar v k) = \frac{\Omega_d}{(2\pi)^d}\, \frac{\varepsilon^{d-1}}{(\hbar v)^d} \ .$$

(b) The condition for $T = T_c$ is to write $n = n(T_c, \mu = 0)$:

$$n = \int_0^\infty \! d\varepsilon\, \frac{g(\varepsilon)}{e^{\varepsilon/k_B T_c} - 1} = \frac{1}{2\pi^2(\hbar v)^3}\int_0^\infty \! d\varepsilon\, \frac{\varepsilon^2}{e^{\varepsilon/k_B T_c} - 1} = \frac{\zeta(3)}{\pi^2}\left(\frac{k_B T_c}{\hbar v}\right)^3 .$$

Thus,

$$k_B T_c = \left(\frac{\pi^2}{\zeta(3)}\right)^{1/3} \hbar v\, n^{1/3} \ .$$

(c) For $T < T_c$, we have

$$n = n_0 + \frac{\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar v}\right)^3 .$$

Thus,

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_c(n)}\right)^3 .$$

(d) In $d = 2$ we have

$$n = \frac{1}{2\pi(\hbar v)^2}\int_0^\infty \! d\varepsilon\, \frac{\varepsilon}{e^{\varepsilon/k_B T_c} - 1} = \frac{\zeta(2)}{2\pi}\left(\frac{k_B T_c}{\hbar v}\right)^2$$

and hence

$$k_{\mathrm{B}} T_{\mathrm{c}}^{(d=2)} = \hbar v \sqrt{\frac{2\pi n}{\zeta(2)}} \,.$$

(2) Using the argument we used in class and in §5.4.3 of the notes, predict the surface temperatures of the remaining planets in our solar system. In each case, compare your answers with the results below. In cases where there are discrepancies, try to come up with a convincing excuse.

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
a (10^8 km)	0.576	1.08	1.50	2.28	7.78	14.3	28.7	45.0	59.1
$T_{\mathrm{surf}}^{\mathrm{obs}}$ (K)	340*	$735^{\dagger}$	$288^{\ddagger}$	210	112	84	53	55	44

Table 1: Relevant planetary data. Observed temperatures are averages. *mean equatorial temperature, †mean temperature below cloud cover. **I say Pluto is a planet.**

Solution :

According to the derivation in the notes, we have

$$T = \left(\frac{R_{\odot}}{2a}\right)^{1/2} T_{\odot} \,,$$

where $R_{\odot} = 6.96 \times 10^5$ km and $T_{\odot} = 5780$ K. From this equation and the reported values for a for each planet, we obtain the following table:

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
a (10^8 km)	0.576	1.08	1.50	2.28	7.78	14.3	28.7	45.0	59.1
$T_{\mathrm{surf}}^{\mathrm{obs}}$ (K)	340*	$735^{\dagger}$	$288^{\ddagger}$	210	112	84	53	55	44
$T_{\mathrm{surf}}^{\mathrm{pred}}$ (K)	448	327	278	226	122	89.1	63.6	50.8	44.3

Table 2: Comparison of observed and predicted planetary surface temperatures. Observed temperatures are averages. *mean equatorial temperature, †mean temperature below cloud cover.

Note that we have included Pluto, because since my childhood Pluto has always been the ninth planet to me. We see that our simple formula works out quite well except for Mercury and Venus. Mercury, being so close to the sun, has enormous temperature fluctuations as a function of location. Venus has a whopping greenhouse effect.

(3) In §5.4.4 of the lecture notes we derived the spectral energy density $\rho_\varepsilon(\nu, T)$ for a three-dimensional blackbody. We found that it was peaked at a frequency $\nu^* = s^* k_B T/h$ where $s^* = 2.83144$ extremizes the function $s^3/(e^s - 1)$. Consider instead the function $\tilde{\rho}_\varepsilon(\lambda, T)$ as a function of wavelength λ and temperature T, where $\lambda = c/\nu$. To relate $\rho_\varepsilon(\nu, T)$ and $\tilde{\rho}_\varepsilon(\lambda, T)$, set the fraction of energy of EM radiation between frequencies ν and $\nu + d\nu$ equal to the fraction of energy between wavelengths λ and $\lambda + d\lambda$. Show that this is maximized at a wavelength $\lambda^* = t^* hc/k_B T$, where t^* is a constant. Find t^* numerically. Is $t^* = 1/s^*$? Why or why not?

Solution:

We must have

$$\tilde{\rho}_\varepsilon(\lambda, T) = \rho_\varepsilon(\nu, T) \left| \frac{d\nu}{d\lambda} \right| = \frac{c}{\lambda^2}\, \rho_\varepsilon(\nu, T)$$
$$= \frac{15}{\pi^4} \frac{k_B T}{hc} \frac{(hc/\lambda k_B T)^5}{e^{hc/\lambda k_B T} - 1} \equiv \frac{15}{\pi^4} \frac{k_B T}{hc} \frac{(\lambda_T/\lambda)^5}{e^{\lambda_T/\lambda} - 1} \quad ,$$

where $\lambda_T \equiv hc/k_B T$ is not to be confused with the thermal de Broglie wavelength for a massive particle. The maximum value occurs for $\lambda^*(T) = u\, k_B T$ where

$$\frac{d}{du}\left(\frac{u^5}{e^u - 1}\right) = 0 \qquad \Rightarrow \qquad u = \frac{u}{1 - e^{-u}} = 5 \qquad \Rightarrow \qquad u = 4.9651 \quad .$$

Thus $\lambda^* = t^* ch/k_B T$ where $t^* = 1/u^* = 0.2014$. Note that $\lambda^*(T) \neq c/\nu^* = 0.3544\, hc/k_B T$. This is because the spectral density $\tilde{\rho}_\varepsilon(\lambda, T)$ is given by $\tilde{\rho}_\varepsilon(\lambda, T) = (c/\lambda^2)\, \rho_\varepsilon(\nu = c/\lambda, T)$ and so the stationary point for λ is obtained by extremizing a different function.

(4) A nonrelativistic Bose gas consists of ballistic particles of spin $S = 1$. Each boson has mass m and magnetic moment μ_0. A gas of these particles is placed in an external field H.

(a) What is the relationship of the Bose condensation temperature $T_c(H)$ to $T_c(H = 0)$ when $\mu_0 H \gg k_B T$?

(b) Find the magnetization M for $T < T_c$ when $\mu_0 H \gg k_B T$. Calculate through order $\exp(-\mu_0 H/k_B T)$.

Solution :

(a) The number density of bosons is given by

$$n(T, z) = \lambda_T^{-3} \Big\{ \mathrm{Li}_{3/2}\big(z\, e^{\mu_0 H/k_B T}\big) + \mathrm{Li}_{3/2}\big(z\big) + \mathrm{Li}_{3/2}\big(z\, e^{-\mu_0 H/k_B T}\big) \Big\} \quad .$$

The argument of $\mathrm{Li}_z(z)$ cannot exceed unity, thus Bose condensation occurs for $z = \exp(-\mu_0 H/k_B T)$ (assuming $H > 0$). Thus, the condition for Bose condensation is given by

$$n\lambda_{T_c}^3 = \zeta(3/2) + \mathrm{Li}_{3/2}\big(e^{-\mu_0 H/k_B T_c}\big) + \mathrm{Li}_{3/2}\big(e^{-2\mu_0 H/k_B T_c}\big) \quad .$$

This is a transcendental equation for $T = T_{\rm c}(n,H)$. In the limit $\mu_0 H \gg k_{\rm B} T_{\rm c}$, the second two terms become negligible, since

$$\mathsf{Li}_s(z) = \sum_{j=1}^\infty \frac{z^j}{j^s} \quad .$$

Thus,

$$T_{\rm c}(H\to\infty) = \frac{2\pi\hbar^2}{m}\bigg(\frac{n}{\zeta(3/2)}\bigg)^{2/3} \quad .$$

When $H=0$, we have Thus,

$$T_{\rm c}(H\to 0) = \frac{2\pi\hbar^2}{m}\bigg(\frac{n}{3\,\zeta(3/2)}\bigg)^{2/3} \quad .$$

Thus,

$$\frac{T_{\rm c}(H\to\infty)}{T_{\rm c}(H\to 0)} = 3^{2/3} = 2.08008\ldots$$

(b) For $T < T_{\rm c}(n,H)$ we have

$$n_\sigma = n^0_\sigma + \lambda_T^{-3}\, \mathsf{Li}_{3/2}\big(e^{-(\sigma+1)\mu_0 H/k_{\rm B}T}\big) \quad ,$$

where n^0_σ is the density of condensed particles with spin polarization σ. Condensation occurs only in the $\sigma=-1$ channel, which contains the single particle ground state $|\, \boldsymbol{k}=0, \sigma=-1\,\rangle$. Thus $n^0_0 = n^0_+ = 0$ but $n^0_- \in [0,n]$ in the condensed phase. The magnetization density is

$$M = -\bigg(\frac{\partial\Omega}{\partial H}\bigg)_{T,V,\mu} = \mu_0\big(n_- - n_+\big)$$

The total number density is $n = n_- + n_0 + n_+$, hence

$$M = \mu_0\big(n - n_0 - 2n_+\big)$$

For $T<T_{\rm c}$, we have $z = \exp(-\mu_0 H/k_{\rm B}T)$ and therefore

$$M = \mu_0\, n - \mu_0 \lambda_T^{-3} e^{-\mu_0 H/k_{\rm B}T} + \mathcal{O}\big(e^{-2\mu_0 H/k_{\rm B}T}\big) \quad .$$