PHYSICS 140B W26 : STATISTICAL PHYSICS
MIDTERM EXAM SOLUTIONS

(1) The formula for isothermal compressibility κ_T is

$$\kappa_T(T,n) = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} = n^{-2}\left(\frac{\partial n}{\partial \mu}\right)_T \quad .$$

(a) Consider an ideal Bose gas of ballistic particles in $d=3$ dimensions, In the condensed phase, *i.e.* for $T < T_c(n)$, the compressibility diverges: $\kappa_T = +\infty$. Why is this the case?

(b) Obtain an expression for $\kappa_T(T,z)$ when $T > T_c(n)$.

(c) Show that $\kappa_T = 1/nk_BT$ in the low density limit. *Hint:* This involves taking the $z \to 0$ limit of your expression from part (b).

Solution :

(a) Since $\mu = 0^-$ is fixed in the condensed phase, $d\mu = 0$ and therefore $\kappa_T = +\infty$ throughout. This is an artifact of the noninteracting idealization: particles in the condensate occupy no volume!

(b) For $T > T_c(n)$ we have

$$\begin{aligned} dn\big|_T &= C\,\Gamma(r)\,(k_BT)^r\, d\mathrm{Li}_r(z) = C\,\Gamma(r)\,(k_BT)^r\, z\,\frac{d\mathrm{Li}_r(z)}{dz}\cdot\frac{dz}{z} \\ &= C\,\Gamma(r)\,(k_BT)^r\,\mathrm{Li}_{r-1}(z)\, d\log z = C\,\Gamma(r)\,(k_BT)^{r-1}\, d\mu \end{aligned}$$

and thus, dividing by $n = C\,\Gamma(r)(k_BT)^r\,\mathrm{Li}_r(z)$ we obtain

$$\kappa_T(T,z) = n^{-2}\left(\frac{\partial n}{\partial \mu}\right)_T = \frac{1}{C\,\Gamma(r)\,(k_BT)^{r+1}}\cdot\frac{\mathrm{Li}_{r-1}(z)}{\big[\mathrm{Li}_r(z)\big]^2} = \frac{\lambda_T^3}{k_BT}\cdot\frac{\mathrm{Li}_{1/2}(z)}{\big[\mathrm{Li}_{3/2}(z)\big]^2} \quad .$$

Note that this diverges as $z \to 1^-$. *i.e.* as the number density is increased to its critical value $n_c(T)$ at fixed temperature T.

(c) As $z \to 0$ we have $\mathrm{Li}_t(z) = z + \mathcal{O}(z^2)$ for all index values r, hence

$$\kappa_T = \frac{1}{k_BT}\cdot\frac{1}{z\lambda_T^{-3}} = \frac{1}{nk_BT} \quad .$$

(2) The dispersion relation for a relativistic massive particle is

$$\varepsilon(\boldsymbol{k}) = \sqrt{(\hbar ck)^2 + (mc^2)^2} = \hbar c\sqrt{k^2 + k_C^2} \quad ,$$

where $k_C = mc/\hbar$ is the Compton wavevector.

(a) Find the single particle density of states $g(\varepsilon)$ in $d = 3$ dimensions, assuming these particles are $S = \frac{1}{2}$ fermions. Sketch $g(\varepsilon)$ *versus* ε for $\varepsilon \in \mathbb{R}$.

(b) What is the Fermi wavevector $k_{\mathrm{F}}(n)$ as a function of the number density?

(c) Find the density of states at the Fermi energy $g(\varepsilon_{\mathrm{F}})$ in terms of $k_{\mathrm{F}}(n)$.

Solution :

(a) We have an internal degeneracy of $\mathsf{g} = 2S + 1 = 2$, hence

$$g(\varepsilon) = \frac{1}{\pi^2} = \frac{1}{\pi^2} \frac{k^2}{d\varepsilon/dk} = \frac{\varepsilon\,(\varepsilon^2 - m^2c^4)^{1/2}}{\pi^2(\hbar c)^3}$$

(b) We have the usual result:

$$n = 2 \cdot \frac{1}{(2\pi)^3} \cdot \tfrac{4}{3}\pi k_{\mathrm{F}}^3 \quad \Rightarrow \quad k_{\mathrm{F}}(n) = (3\pi^2 n)^{1/3} \quad .$$

(c) Substituting,

$$g(\varepsilon_{\mathrm{F}}) = \frac{1}{\pi^2 \hbar c} k_{\mathrm{F}} \sqrt{k_{\mathrm{F}}^2 + k_{\mathrm{C}}^2} \quad .$$

Potentially useful formulae:

$$\mathsf{Li}_t(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^t} \qquad \mathsf{Li}_t(1) = \zeta(t) \qquad z\,\frac{d}{dz}\,\mathsf{Li}_t(z) = \mathsf{Li}_{t-1}(z)$$

$$g(\varepsilon)\,d\varepsilon = \frac{\mathsf{g}\,\Omega_d}{(2\pi)^2}\,k^{d-1}\,dk \qquad g(\varepsilon) = C\varepsilon^{r-1}\,\Theta(\varepsilon) \qquad r = d/\theta \quad , \quad z = \exp(\mu/k_{\mathrm{B}}T)$$

$$\beta = \frac{1}{k_{\mathrm{B}}T} \qquad \int\limits_0^\infty d\varepsilon\,\frac{\varepsilon^{t-1}}{z^{-1}\,e^{\beta\varepsilon} - 1} = \Gamma(t)\,(k_{\mathrm{B}}T)^t\,\mathsf{Li}_t(z)$$

$$n_{\mathrm{c}}(T) = C\,\Gamma(r)\,\zeta(r)\,(k_{\mathrm{B}}T)^r \qquad p_{\mathrm{c}}(T) = C\,\Gamma(r)\,\zeta(r+1)\,(k_{\mathrm{B}}T)^{r+1}$$

$$n(T < T_{\mathrm{c}}, n_0) = n_0 + n_{\mathrm{c}}(T) \qquad n(T > T_{\mathrm{c}}, \mu) = C\,\Gamma(r)(k_{\mathrm{B}}T)^r\,\mathsf{Li}_r(z)$$

$$p(T < T_{\mathrm{c}}, n_0) = p_{\mathrm{c}}(T) \qquad p(T > T_{\mathrm{c}}, \mu) = C\,\Gamma(r)(k_{\mathrm{B}}T)^{r+1}\,\mathsf{Li}_{r+1}(z)$$

For ballistic particles, $C\,\Gamma(r)(k_{\mathrm{B}}T)^r = \lambda_T^{-d}$ with $\lambda_T = \sqrt{2\pi\hbar^2/mk_{\mathrm{B}}T}$.

$$f_T(\omega) = \frac{1}{e^{\beta\omega} + 1} \qquad f_{T=0^+}(\omega) = \Theta(-\omega)$$

$$\widetilde{\mathcal{I}}(T,\mu) = \int\limits_{-\infty}^{\infty} d\varepsilon\; f_T(\varepsilon - \mu)\,\phi(\varepsilon) = \int\limits_{-\infty}^{\mu} d\varepsilon\,\phi(\varepsilon) + \frac{\pi^2}{6}\,(k_{\mathrm{B}}T)^2\,\phi'(\mu) + \frac{7\pi^4}{360}\,(k_{\mathrm{B}}T)^4\,\phi'''(\mu) + \ldots$$