PHYSICS 140B W26 : STATISTICAL PHYSICS
MIDTERM EXAM

(1) The formula for isothermal compressibility κ_T is

$$\kappa_T(T,n) = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} = n^{-2}\left(\frac{\partial n}{\partial \mu}\right)_T \quad .$$

(a) Consider an ideal Bose gas of ballistic particles in $d=3$ dimensions, In the condensed phase, *i.e.* for $T < T_\mathrm{c}(n)$, the compressibility diverges: $\kappa_T = +\infty$. Why is this the case?

(b) Obtain an expression for $\kappa_T(T,z)$ when $T > T_\mathrm{c}(n)$.

(c) Show that $\kappa_T = 1/nk_\mathrm{B}T$ in the low density limit. *Hint:* This involves taking the $z \to 0$ limit of your expression from part (b).

(2) The dispersion relation for a relativistic massive particle is

$$\varepsilon(\boldsymbol{k}) = \sqrt{(\hbar c k)^2 + (mc^2)^2} = \hbar c\sqrt{k^2 + k_\mathrm{C}^2} \quad ,$$

where $k_\mathrm{C} = mc/\hbar$ is the Compton wavevector.

(a) Find the single particle density of states $g(\varepsilon)$ in $d=3$ dimensions, assuming these particles are $S=\frac{1}{2}$ fermions. Sketch $g(\varepsilon)$ *versus* ε for $\varepsilon \in \mathbb{R}$.

(b) What is the Fermi wavevector $k_\mathrm{F}(n)$ as a function of the number density?

(c) Find the density of states at the Fermi energy $g(\varepsilon_\mathrm{F})$ in terms of $k_\mathrm{F}(n)$.

Potentially useful formulae:

$$\mathsf{Li}_t(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^t} \qquad \mathsf{Li}_t(1) = \zeta(t) \qquad z\,\frac{d}{dz}\,\mathsf{Li}_t(z) = \mathsf{Li}_{t-1}(z)$$

$$g(\varepsilon)\,d\varepsilon = \frac{\mathsf{g}\,\Omega_d}{(2\pi)^2}\,k^{d-1}\,dk \qquad g(\varepsilon) = C\varepsilon^{r-1}\,\Theta(\varepsilon) \qquad r = d/\theta \quad , \quad z = \exp(\mu/k_{\rm B}T)$$

$$\beta = \frac{1}{k_{\rm B}T} \qquad \int\limits_0^\infty \! d\varepsilon\,\frac{\varepsilon^{t-1}}{z^{-1}\,e^{\beta\varepsilon}-1} = \Gamma(t)\,(k_{\rm B}T)^t\,\mathsf{Li}_t(z)$$

$$n_{\rm c}(T) = C\,\Gamma(r)\,\zeta(r)\,(k_{\rm B}T)^r \qquad p_{\rm c}(T) = C\,\Gamma(r)\,\zeta(r+1)\,(k_{\rm B}T)^{r+1}$$

$$n(T<T_{\rm c},n_0) = n_0 + n_{\rm c}(T) \qquad n(T>T_{\rm c},\mu) = C\,\Gamma(r)(k_{\rm B}T)^r\,\mathsf{Li}_r(z)$$

$$p(T<T_{\rm c},n_0) = p_{\rm c}(T) \qquad p(T>T_{\rm c},\mu) = C\,\Gamma(r)(k_{\rm B}T)^{r+1}\,\mathsf{Li}_{r+1}(z)$$

For ballistic particles, $C\,\Gamma(r)(k_{\rm B}T)^r = \lambda_T^{-d}$ with $\lambda_T = \sqrt{2\pi\hbar^2/mk_{\rm B}T}$.

$$f_T(\omega) = \frac{1}{e^{\beta\omega}+1} \qquad f_{T=0^+}(\omega) = \Theta(-\omega)$$

$$\widetilde{\mathcal{I}}(T,\mu) = \int\limits_{-\infty}^{\infty} \! d\varepsilon\; f_T(\varepsilon-\mu)\,\phi(\varepsilon) = \int\limits_{-\infty}^{\mu} \! d\varepsilon\,\phi(\varepsilon) + \frac{\pi^2}{6}\,(k_{\rm B}T)^2\,\phi'(\mu) + \frac{7\pi^4}{360}\,(k_{\rm B}T)^4\,\phi'''(\mu) + \ldots$$