PHYSICS 140B W26 : STATISTICAL PHYSICS
HW ASSIGNMENT #6

(1) Consider the free energy

$$f(\theta, m) = f_0 + \tfrac{1}{2}am^2 + \tfrac{1}{4}bm^4 + \tfrac{1}{8}dm^8$$

with $d > 0$. Note there is an octic term but no sextic term. Derive results corresponding to those in fig. 7.16 of the lecture notes. Find the equation of the first order line in the $(a/d, b/d)$ plane. Also identify the region in parameter space where there exist metastable local minima in the free energy (curve E in fig. 7.16).

(2) Consider the four-state clock model,

$$\hat{H} = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\,\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j - H\hat{\boldsymbol{x}} \cdot \sum_{i=1}^{N} \hat{\boldsymbol{n}}_i \quad ,$$

where at each site $\hat{\boldsymbol{n}}_i \in \{\pm\hat{\boldsymbol{x}}, \pm\hat{\boldsymbol{y}}\}$. Work out the variational density matrix mean field theory, taking $\varrho_{\mathsf{var}}(\hat{\boldsymbol{n}}_1, \ldots, \hat{\boldsymbol{n}}_N) = \prod_{i=1}^{N} \varrho(\hat{\boldsymbol{n}}_i)$ with the single site variational density matrix

$$\varrho(\hat{\boldsymbol{n}}) = \tfrac{1}{2}(1+u-v)\,\delta_{\hat{\boldsymbol{n}},\hat{\boldsymbol{x}}} + \tfrac{1}{2}(1-u-v)\,\delta_{\hat{\boldsymbol{n}},-\hat{\boldsymbol{x}}} + \tfrac{1}{2}\,v\,\delta_{\hat{\boldsymbol{n}},\hat{\boldsymbol{y}}} + \tfrac{1}{2}\,v\,\delta_{\hat{\boldsymbol{n}},-\hat{\boldsymbol{y}}} \quad .$$

(a) What is the global symmetry group for this Hamiltonian when $H = 0$? (Give the name of the group, or express how the symmetry operations act in words/equations.)

(b) Evaluate $E = \mathsf{Tr}(\varrho_{\mathsf{var}}\,\hat{H})$.

(c) Evaluate $S = -k_{\mathrm{B}}\,\mathsf{Tr}(\varrho_{\mathsf{var}} \log \varrho_{\mathsf{var}})$.

(d) Adimensionalize by writing $\theta = k_{\mathrm{B}}T/\hat{J}(0)$ and $h = H/\hat{J}(0)$, where z is the lattice coordination number. Find $f(u, v, \theta, h) = F/N\hat{J}(0)$.

(e) Find all the mean field equations.

(f) Show that the solution to the mean field equations requires that $v = v(u)$ is a function of u alone and find $v(u)$

(g) Substituting $v(u)$, find $f(u, \theta, h) = f(u, v(u), \theta, h)$.

(h) Expand $f(u, \theta, h)$ as a power series in u up to order u^6 and identify the critical temperature θ_{c} and the order of the phase transition.