

PHYSICS 140B W26 : STATISTICAL PHYSICS
HW ASSIGNMENT #5

(1) The Dieterici equation of state is

$$p(v - b) = RT \exp\left(-\frac{a}{vRT}\right).$$

(a) Find the critical point (p_c, v_c, T_c) for this equation of state

(a) Writing $\bar{p} = p/p_c$, $\bar{v} = v/v_c$, and $\bar{T} = T/T_c$, rewrite the equation of state in the form $\bar{p} = \bar{p}(\bar{v}, \bar{T})$.

(2) Consider an Ising ferromagnet where the nearest neighbor exchange temperature is $J_{\text{NN}}/k_{\text{B}} = 50$ K and the next nearest neighbor exchange temperature is $J_{\text{NNN}}/k_{\text{B}} = 10$ K. What is the mean field transition temperature T_c if the lattice is:

- (a) square
- (b) honeycomb
- (c) triangular
- (d) simple cubic
- (e) body centered cubic

Hint : As an intermediate step, you might want to show that the mean field transition temperature is given by

$$k_{\text{B}}T_c^{\text{MF}} = z_1 J_{\text{NN}} + z_2 J_{\text{NNN}},$$

where z_1 and z_2 are the number of nearest neighbors and next-nearest neighbors of a given lattice site, respectively.

(3) Consider a three state Ising model,

$$\hat{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i,$$

where $S_i \in \{-1, 0, +1\}$.

(a) Writing $S_i = m + \delta S_i$ and ignoring terms quadratic in the fluctuations, derive the mean field Hamiltonian H_{MF} .

(b) Find the dimensionless mean field free energy density, $f = F_{\text{MF}}/NzJ$, where z is the lattice coordination number. You should define the dimensionless temperature $\theta \equiv k_{\text{B}}T/zJ$ and the dimensionless field $h \equiv H/zJ$.

(c) Find the self-consistency equation for $m = \langle S_i \rangle$ and show that this agrees with the condition $\partial f / \partial m = 0$.

(d) Expand $f(m)$ to fourth order in m and first order in h .

(e) Find the critical temperature θ_c .

(d) Find $m(\theta_c, h)$.

(4) For the O(3) Heisenberg ferromagnet,

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j ,$$

find the mean field transition temperature T_c^{MF} . Here, each $\hat{\mathbf{n}}_i$ is a three-dimensional unit vector, which can be parameterized using the usual polar and azimuthal angles:

$$\hat{\mathbf{n}}_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i) .$$

The thermodynamic trace is defined as

$$\text{Tr} A(\hat{\mathbf{n}}_1, \dots, \hat{\mathbf{n}}_N) = \int \prod_{i=1}^N \frac{d\hat{\mathbf{n}}_i}{4\pi} A(\hat{\mathbf{n}}_1, \dots, \hat{\mathbf{n}}_N) ,$$

where

$$d\hat{\mathbf{n}}_i = \sin \theta_i d\theta_i d\phi_i .$$

Hint : Your mean field Ansatz will look like $\hat{\mathbf{n}}_i = \mathbf{m} + \delta\mathbf{n}_i$, where $\mathbf{m} = \langle \mathbf{n}_i \rangle$. You'll want to ignore terms in the Hamiltonian which are quadratic in fluctuations, i.e. $\delta\mathbf{n}_i \cdot \delta\mathbf{n}_j$. You can, without loss of generality, assume \mathbf{m} to lie in the \hat{z} direction.

(5) The spin lattice Hamiltonian for the three state (\mathbb{Z}_3) clock model is written

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j ,$$

where each local unit vector $\hat{\mathbf{n}}_i$ is a planar spin which can take one of three possible values:

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_1 \quad , \quad \hat{\mathbf{n}} = -\frac{1}{2}\hat{\mathbf{e}}_1 + \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_2 \quad , \quad \hat{\mathbf{n}} = -\frac{1}{2}\hat{\mathbf{e}}_1 - \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_2 \quad .$$

Note that the *internal space* in which each unit vector $\hat{\mathbf{n}}_i$ exists is distinct from the physical Euclidean space in which the lattice points reside.

(a) Consider the clock model on a lattice of coordination number z . Make the mean field assumption $\langle \hat{\mathbf{n}}_i \rangle = m \hat{\mathbf{e}}_1$. Expanding the Hamiltonian to linear order in the fluctuations, derive the mean field Hamiltonian for this model \hat{H}_{MF} .

(b) Rescaling $\theta = k_B T / zJ$ and $f = F / NzJ$, where F is the Helmholtz free energy and N is the number of sites, find $f(m, \theta)$.

(c) Is the transition second order or first order? Why?

(d) Find the equations which determine the critical temperature θ_c .

(e) Show that this model is equivalent to the three state Potts model. Is the \mathbb{Z}_4 clock model equivalent to the four state Potts model? Why or why not?