PHYSICS 140B W26 : STATISTICAL PHYSICS
HW ASSIGNMENT #4

(1) For the Mayer cluster expansion, write down all possible unlabeled connected subgraphs γ which contain four vertices. For your favorite of these animals, identify its symmetry factor s_γ, and write down the corresponding expression for the cluster integral b_γ. For example, for the $\Box$ diagram with four vertices the symmetry factor is $s_\Box = 8$ and the cluster integral is

$$b_\Box = \frac{1}{8V\lambda_T^{3d}} \int\! d^d\!r_1 \int\! d^d\!r_2 \int\! d^d\!r_3 \int\! d^d\!r_4 \, f(r_{12})\, f(r_{23})\, f(r_{34})\, f(r_{14})$$
$$= \frac{1}{8} \int\! \frac{d^d\!r_1}{\lambda_T^d} \int\! \frac{d^d\!r_2}{\lambda_T^d} \int\! \frac{d^d\!r_3}{\lambda_T^d} \, f(r_{12})\, f(r_{23})\, f(r_1)\, f(r_3) \ .$$

(You'll have to choose a favorite other than $\Box$.) If you're really energetic, compute s_γ and b_γ for all of the animals with four vertices.

(2) An ionic solution of dielectric constant ϵ and mean ionic density n fills a grounded conducting sphere of radius R. A charge Q lies at the center of the sphere. Calculate the ionic charge density as a function of the radial coordinate r, assuming $Q/r \ll k_\mathrm{B}T$.

(3) The Blume-Capel model is a spin-1 version of the Ising model, with Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i \, S_j - \Delta \sum_i S_i^2 \ ,$$

where $S_i \in \{-1\,,\,0\,,\,+1\}$ and where the first sum is over all links of a lattice and the second sum is over all sites. It has been used to describe magnetic solids containing vacancies ($S = 0$ for a vacancy) as well as phase separation in $^4\mathrm{He} - {}^3\mathrm{He}$ mixtures ($S = 0$ for a ^{4}He atom). For parts (b), (c), and (d) you should work in the thermodynamic limit. The eigenvalues and eigenvectors are such that it would shorten your effort considerably to use a program like `Mathematica` to obtain them.

(a) Find the transfer matrix for the $d = 1$ Blume-Capel model.

(b) Find the free energy $F(T, \Delta, N)$.

(c) Find the density of $S = 0$ sites as a function of T and Δ.

(d) *Exciting!* Find the correlation function $\langle \, S_j \, S_{j+n} \, \rangle$.