PHYSICS 140B W26 : STATISTICAL PHYSICS
FINAL EXAMINATION

(1) Consider the 4-state clock model on a one-dimensional ring of N sites. The Hamiltonian is given by

$$\hat{H} = -J\sum_{j=1}^{N} \hat{\boldsymbol{n}}_j \cdot \hat{\boldsymbol{n}}_{j+1} - H\hat{\boldsymbol{x}} \cdot \sum_{j=1}^{N} \hat{\boldsymbol{n}}_j \quad .$$

Each unit vector $\hat{\boldsymbol{n}}_j$ can take one of four possible values: $\hat{\boldsymbol{n}}_j \in \{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, -\hat{\boldsymbol{x}}, -\hat{\boldsymbol{y}}\}$.

(a) Find an expression for the symmetric transfer matrix $T_{\hat{\boldsymbol{n}},\hat{\boldsymbol{n}}'}$ and express it in matrix form. `[10 points]`

(b) Find all the eigenvalues of the transfer matrix when $H = 0$. `[8 points]`

Big Hint: For the matrix

$$T = \begin{pmatrix} r & 1 & r^{-1} & 1 \\ 1 & r & 1 & r^{-1} \\ r^{-1} & 1 & r & 1 \\ 1 & r^{-1} & 1 & r \end{pmatrix}$$

the four (unnormalized) eigenvectors may be taken to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad .$$

(c) Find the free energy per site in the thermodynamic limit $N \to \infty$ when $H = 0$. Interpret the $T \to \infty$ limit of your result. `[7 points]`

(2) Consider the modified van der Waals equation of state

$$\left(p + \frac{a}{(v+b)^2}\right)(v - b) = RT \quad ,$$

where $v = N_{\rm A} V/N$ is the molar volume. Find the critical point $(v_{\rm c}, T_{\rm c}, p_{\rm c})$.

(a) Find $v_{\rm c}$. `[8 points]`

(b) Find $T_{\rm c}$. `[8 points]`

(c) Find $p_{\rm c}$. `[9 points]`

(3) The Blume-Capel Hamiltonian is

$$\hat{H} = -\tfrac{1}{2}\sum_{i,j=1}^{N} J_{ij}\, S_i\, S_j + \Delta \sum_{i=1}^{N} S_i^2 \quad .$$

The spin variables S_i range over the values $\{-1\,,\,0\,,\,+1\}$, so this is an extension of the $S=1$ Ising model. All the diagonal entries of the interaction matrix J are set to zero: $J_{ii}\equiv 0$.

(a) *In the first term only*, make the mean field *Ansatz*, writing $S_i = m + \delta S_i$ and derive the mean field Hamiltonian $\hat{H}_{\mathsf{MF}}$. The term proportional to Δ enters $\hat{H}_{\mathsf{MF}}$ as it appears above. You may assume $J_{ij} = J(|\boldsymbol{R}_i - \boldsymbol{R}_j|)$ and write $\hat{J}(0)\equiv\sum_{\boldsymbol{R}} J(\boldsymbol{R})$ as we have done before, where $\boldsymbol{R}$ is a lattice translation symmetry vector. (If the interactions are only between nearest-neighbor sites, then $\hat{J}(0) = zJ$ where z is the number of nearest neighbors.) Again, *only neglect fluctuations in the first term – you will treat the second term exactly.* [7 points]

(b) Derive the mean field free energy $F(m,T,\Delta,N)$. [6 points]

(c) Write the dimensionless mean field free energy per site $f(m,\theta,\delta) = F/N\hat{J}(0)$, where $\theta = k_{\rm B}T/\hat{J}(0)$, $f = F/N\hat{J}(0)$, and $\delta = \Delta/\hat{J}(0)$. What is the self-consistent mean field equation? [6 points]

(d) If an external field term $-H\sum_{i=1}^N S_i$ is included in the Hamiltonian, what is the free energy $f(m,\theta,h,\delta)$, where $h = H/\hat{J}(0)$, and what is the mean field equation? [6 points]

(4) Consider a monatomic ideal gas in the presence of a temperature gradient $\boldsymbol{\nabla}T$. Answer the following questions within the framework of the relaxation time approximation to the Boltzmann equation.

(a) Compute the particle current $\boldsymbol{j}$ and show that it vanishes. [8 points]

(b) Compute the 'energy squared' current,

$$\boldsymbol{j}_{\varepsilon^2} = \int\! d^3\!p\,\varepsilon^2 \boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{p},t) \quad .$$

Hints: Equilibrium averages of $\phi(\boldsymbol{v})$:

$$\int\! d^3\!p\, f^0(\boldsymbol{p})\,\phi(\boldsymbol{v}) = n\!\int\! d^3\!v P(\boldsymbol{v})\,\phi(\boldsymbol{v}) \quad , \quad P(\boldsymbol{v}) = \left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2} \exp\left(-\frac{m\boldsymbol{v}^2}{2k_{\rm B}T}\right)$$

For the Maxwell distribution of energies:

$$\big\langle \varepsilon^\alpha \big\rangle = \tfrac{2}{\sqrt{\pi}}\,\Gamma\big(\alpha+\tfrac{3}{2}\big)\,(k_{\rm B}T)^\alpha \quad .$$

Recall $\Gamma(1/2)=\sqrt{\pi}$ and $z\,\Gamma(z) = \Gamma(z+1)$. [8 points]

(c) Compute the scalar momentum current,

$$\boldsymbol{j}_p = \int\! d^3\!p\, p\,\boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{p},t) \quad .$$

[9 points]