Physics 4B Spring 2026
Solutions to Week 7 Additional Homework Problems

1. **Problem 3**. In a gas of helium atoms ($m_{\text{He}} = 6.646 \times 10^{-27}$ kg) at 300 K, there are 10,000 atoms that have velocity in the x direction between -1 m/s and 1 m/s. Approximately how many atoms have velocity in the y direction between 100 m/s and 101 m/s?

   Solution: This means

$$10000 = N \cdot \int_{-1}^{1} dv_x f(v_x) = N \left(\frac{m}{2\pi kT}\right)^{1/2} \int_{-1}^{1} dv_x \exp\left(-\frac{mv_x^2}{2kT}\right) \tag{1}$$

   We can evaluate the integral to determine N, the total number of helium atoms, using the mass and temperature given to us. The number of atoms that have $v_y \in [100, 101]$ m/s would be

$$\begin{aligned} N \cdot \int_{100}^{101} dv_y f(v_y) &= N \left(\frac{m}{2\pi kT}\right)^{1/2} \int_{100}^{101} dv_y \exp\left(-\frac{mv_y^2}{2kT}\right) \\ &= 10000 \cdot \frac{\int_{100}^{101} dv_y \exp\left(-\frac{mv_y^2}{2kT}\right)}{\int_{-1}^{1} dv_x \exp\left(-\frac{mv_x^2}{2kT}\right)} \end{aligned} \tag{2}$$

   Now, since the intervals we are considering ($v_x \in [-1, 1]$ m/s and $v_y \in [100, 101]$ m/s) are narrow, we can approximate the above integrals by taking the integrand to be roughly constant over the interval:

$$\begin{aligned} 10000 \cdot \frac{\int_{100}^{101} dv_y \exp\left(-\frac{mv_y^2}{2kT}\right)}{\int_{-1}^{1} dv_x \exp\left(-\frac{mv_x^2}{2kT}\right)} &\approx 10000 \cdot \frac{\Delta v_y \cdot \exp\left(-\frac{m\bar{v}_y^2}{2kT}\right)}{\Delta v_x \cdot \exp\left(-\frac{m\bar{v}_x^2}{2kT}\right)} \\ &= 10000 \cdot \frac{1}{2} \exp\left(\frac{m(\bar{v}_x^2 - \bar{v}_y^2)}{2kT}\right) \\ &\approx 5000 \end{aligned} \tag{3}$$

2. **Problem 5**. For a gas at temperature T, the most probable speed of the molecules is v_0. For what temperature is the average speed v_0?

   Solution: The most probable speed as a function of T is given by

$$v_0(T) = \sqrt{\frac{2kT}{m}} \tag{4}$$

whereas the average speed as a function of T is given by

$$\langle v\rangle(T)=\sqrt{\frac{8kT}{\pi m}} \tag{5}$$

We want to solve for the temperature T' such that $\langle v\rangle(T')=v_0(T)$, i.e.

$$\sqrt{\frac{8kT'}{\pi m}}=\sqrt{\frac{2kT}{m}} \tag{6}$$

With some algebra, we find

$$T'=\frac{\pi}{4}T \tag{7}$$

which is slightly smaller than T.

3. **Problem 7**. Compute the average value of $|v_x|$ for an ideal gas. How does it compare to the average speed $\langle v\rangle$?

   Solution: This requires us to look at the distribution for v_x only, which is:

$$f(v_x)=\left(\frac{m}{2\pi kT}\right)^{1/2}\exp\left(-\frac{mv_x^2}{2kT}\right) \tag{8}$$

   Following the same procedure as in our calculations for $\langle v\rangle$ and $\langle v^2\rangle$, we have

$$\begin{aligned}\langle|v_x|\rangle &= \int_{-\infty}^{\infty}dv_x\ |v_x|f(v_x)=\left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{\infty}dv_x\ |v_x|\exp\left(-\frac{mv_x^2}{2kT}\right)\\ &=\left(\frac{a}{\pi}\right)^{1/2}2\int_0^{\infty}dv_x\ v_xe^{-av_x^2}=\left(\frac{a}{\pi}\right)^{1/2}a^{-1}=\sqrt{\frac{2kT}{\pi m}}=\frac{1}{2}\langle v\rangle\end{aligned} \tag{9}$$

4. **Problem 8**. Assume there is a very small hole in the wall of a container that contains an ideal gas. Show that the mean kinetic energy of the molecules that escape is larger than $\frac{3}{2}kT$. What is it?.

   Solution: Inside the container, the average kinetic energy per molecule is $\frac{3}{2}kT$, which is equally split between the 3 directions:

$$\frac{1}{2}m\langle v_x^2\rangle_{\text{inside}}=\frac{1}{2}m\langle v_y^2\rangle_{\text{inside}}=\frac{1}{2}m\langle v_z^2\rangle_{\text{inside}}=\frac{1}{2}kT \tag{10}$$

   However, the molecules that escape from the hole are biased and are not a random sample - particles that have large velocity component perpendicular to the wall with the hole are more likely to escape, so (10) doesn't apply to them. For these particles, we need to make different assumptions to compute the rms velocities $\langle v_i^2 \rangle$ $(i = x, y, z)$.

Let the direction perpendicular to the wall with the hole be z. It would be reasonable to argue that faster velocities in the x, y directions don't make the particles more likely to escape the hole, so the average kinetic energy in these directions for the *escaping particles* should remain the same as those inside the container:

$$\frac{1}{2}m\langle v_x^2 \rangle_{\text{esc}} = \frac{1}{2}m\langle v_y^2 \rangle_{\text{esc}} = \frac{1}{2}kT \tag{11}$$

For the z direction, the probability of a particle hitting the wall is directly proportional to v_z, since a particle with twice the value of v_z would hit the wall twice as often. Also, particles that escape must have $v_z > 0$. Therefore, the distribution for the z-component of the velocity for the *escaping particles* should take the form

$$f_{\text{esc}}(v_z) \propto v_z \exp\left(-\frac{mv_z^2}{2kT}\right) \tag{12}$$

with $v_z \in (0, \infty)$. The normalization factor is determined by the condition $1 = \int_0^\infty dv_z\ f_{\text{esc}}(v_z)$. Using this, we can compute the average kinetic energy in the z direction for the escaped particles (again we define $a \equiv m/2kT$):

$$\begin{aligned}
\frac{1}{2}m\langle v_z^2 \rangle_{\text{esc}} &= \frac{1}{2}m \cdot \frac{\int_0^\infty dv_z\ v_z^2 f_{\text{esc}}(v_z)}{\int_0^\infty dv_z\ f_{\text{esc}}(v_z)} = \frac{1}{2}m \cdot \frac{\int_0^\infty dv_z\ v_z^3 e^{-av_z^2}}{\int_0^\infty dv_z\ v_z e^{-av_z^2}} \\
&= \frac{1}{2}m \cdot \frac{\frac{1}{2}a^{-2}}{\frac{1}{2}a^{-1}} = \frac{1}{2}ma^{-1} = kT
\end{aligned} \tag{13}$$

Therefore, the mean kinetic energy of the escaped particles is

$$\frac{1}{2}m\langle v^2 \rangle_{\text{esc}} = \frac{1}{2}m\left(\langle v_x^2 \rangle_{\text{esc}} + \langle v_y^2 \rangle_{\text{esc}} + \langle v_z^2 \rangle_{\text{esc}}\right) = 2kT > \frac{3}{2}kT \tag{14}$$