Solution to atmosphere problem

If ρ is constant, $P_0 = \rho_0 g H \Rightarrow$

$$H = \frac{P_0}{\rho_0 g} = \frac{101,325}{1.204 \cdot g} = \boxed{8581\,m} \quad \text{and} \quad \boxed{\frac{P_0}{\rho_0} = gH}$$

(b)

$$dP = -\rho(h)\, g\, dh \quad . \quad \text{Assume } P = \alpha \rho \Rightarrow$$

$$\Rightarrow d\rho = -\frac{g}{\alpha}\rho \Rightarrow \frac{d\rho}{\rho} = -\frac{g}{\alpha} \Rightarrow \ln \rho = -\frac{g}{\alpha} h + \text{constant} \Rightarrow$$

$$\Rightarrow \rho = \rho_0 e^{-\frac{g}{\alpha}h} \quad . \quad \text{and } \alpha = \frac{P}{\rho} = \frac{P_0}{\rho_0} = gH \Rightarrow$$

$$\Rightarrow \boxed{\rho(h) = \rho_0 e^{-h/H}}$$

(c) g varies with h: $g(h) = \frac{GM}{(R+h)^2} = \frac{GM}{R^2(1+\frac{h}{R})^2} = \frac{g_0}{(1+\frac{h}{R})^2}$

g_0 is g at sea level $= \frac{GM}{R^2}$

$$g(h) = \frac{g_0}{(1+h/R)^2} \approx g_0\left(1 - \frac{2h}{R}\right) \quad \text{for } h \ll R = 6370\,km$$

differential eq. is now

$$d\rho = -\rho(h)\frac{g(h)\,dh}{\alpha} = -\frac{\rho(h)\,g(h)}{g_0 H}dh = -\frac{\rho\, \cancel{g_0}}{\cancel{g_0} H(1+h/R)^2}dh \Rightarrow$$

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{dh}{H(1+h/R)^2} \approx -\frac{dh}{H}\left(1 - \frac{2h}{R}\right) \Rightarrow$$

$$\ln \rho = -\frac{h}{H} + \frac{h^2}{HR} + \text{Constant} \Rightarrow$$

$$\boxed{\rho(h) = \rho_0 e^{-h/H + h^2/HR}}$$

this assumes $\rho(h=0) = \rho_0$ both for constant g and variable g

Given

$\rho(h) = \rho_o e^{-h/R + h^2/HR}$ versus $\rho(h) = \rho_o e^{-h/R}$

it is clear that $\rho(h)$ became larger at all h when we included the effect of g decreasing with h. That is because we imposed the constraint that $\rho(h=0)$ was the same in both cases.

Instead, let's see what happens if we let g change with h with the constraint that the number of molecules in a column stays same.

Qualitatively it is clear that:

for small h, ρ should decrease, air molecules move up because g is becoming weaker

for large h, ρ should increase because where ρ is very small, molecules that move up from further down increase the density

Where is the crossover?

With the constraint $\int_0^\infty dh\, \rho(h) = \text{constant}$

let's approximate $\rho(h) = \rho_o' e^{-h/R + h^2/HR} \simeq \rho_o' e^{-h/R}\left(1 + \frac{h^2}{HR}\right)$ valid for $h \ll R$

$$\int_0^\infty dh\, \rho_o' e^{-h/R}\left(1 + \frac{h^2}{HR}\right) = \rho_o' R\left(1 + \frac{2H}{R}\right)$$

and $\int_0^h dh\, \rho_o e^{-h/R} = \rho_o R \Rightarrow \rho_o' = \rho_o / (1 + 2H/R)$

So $\boxed{\rho(h) = \frac{\rho_o}{1+2H/R} e^{-h/H + h^2/HR}}$

clearly, for small h, ρ has decreased, for large h it increased.

but this is valid only in regime $h \ll R$, since we did approximations based on that. For say $h/R = 0.1 \Rightarrow h = 637$ km and $H = 8.581$ km, clearly $\frac{h}{H}$ is very small and $\rho(h)$ is essentially zero.

Let's see where the crossover is:

$$\rho(h) = \frac{\rho_0}{1+\frac{2H}{R}} e^{-h/H} e^{h^2/HR} \simeq \frac{\rho_0}{1+\frac{2H}{R}} e^{-h/H}\left(1+\frac{h^2}{HR}\right) = \rho_0 e^{-h/H}$$

$$\Rightarrow \quad 1+\frac{h^2}{HR} = 1+\frac{2H}{R} \Rightarrow h^2 = 2H^2 \Rightarrow \boxed{h=\sqrt{2}H}$$

so $h = \sqrt{2}H = 12.1\,km$

So for $h < 12.1\,km$ ρ ~~increases~~ decreases, for $h > 12.1\,km$ it increases.

At $h = \sqrt{2}H$, density is $\rho = \rho_0 e^{-\sqrt{2}} = 0.24\,\rho_0$

Now let's see what happens if we don't assume $h \ll R$

$$\frac{d\rho}{\rho} = -\frac{dh}{H(1+h/R)^2} \Rightarrow \ln\rho = -\frac{R}{H}\frac{1}{1+h/R} + \text{const} \Rightarrow$$

$$\Rightarrow \boxed{\rho(h) = \rho(0)\, e^{-\frac{h}{H}\frac{1}{1+h/R}}}$$

We haven't assumed $h \ll R$. So in this expression, we can use any h. As $h \to \infty$

$$\rho(h\to\infty) = \rho(0)\, e^{-\frac{h}{H}\frac{1}{1+h/R}} \to \rho(0)e^{-R/H} = \rho(0)e^{-741}$$

So $\rho(h)$ does not go to 0 as $h \to \infty$ $\Rightarrow$ we cannot assume that $\int_0^\infty dh\,\rho(h)$ stays constant when g varies with h, since that gives an atmosphere that extends to infinity.

If these expressions applied to reality, the earth would lose its atmosphere, and we wouldn't be here.

The reason they don't is that we assumed constant temperature throughout when we said $P = \alpha\rho$, in reality that is not so, temperature decreases as h increases (mostly)