Quiz 3 supplement solution

$$\psi(x) = \int_{-\infty}^{\infty} dk\, a(k)\, e^{ikx}$$

$$a(k) = C e^{-x_0^2 (k-k_0)^2 - ikx_0}$$

$x_0 = 6\,nm$, $k_0 = 0.3\,nm^{-1}$

$$\psi(x) = C \int_{-\infty}^{\infty} dk\, e^{-x_0^2 (k-k_0)^2 + ik(x-x_0)}$$

define $\bar{x} = x - x_0$, $\alpha = x_0$. Then

$$\psi(x) = C \int_{-\infty}^{\infty} dk\, e^{-\alpha^2 (k-k_0)^2 + ik\bar{x}}$$

we did integral in class, by completing squares, and found (lecture 12)

$$\psi(x) \propto e^{ik_0 x} e^{-x^2/4\alpha^2}$$

So here:

$$\boxed{\psi(x) = C e^{ik_0(x-x_0)} e^{-(x-x_0)^2/4x_0^2}}$$

(C is not the same as above)

(b) $\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1 \Rightarrow$

$$1 = C^2 \int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/2x_0^2}$$

change variables $x - x_0 \equiv y$

$$1 = C^2 \int_{-\infty}^{\infty} dy\, e^{-y^2/2x_0^2} = \sqrt{\pi \cdot 2x_0^2} = C^2 x_0 \sqrt{2\pi}$$

$$\Rightarrow C^2 = \frac{1}{x_0\sqrt{2\pi}} \Rightarrow \boxed{C = \frac{1}{\sqrt{x_0\sqrt{2\pi}}}}$$

(c) $$\langle x \rangle = \int_{-\infty}^{\infty} dx\, x\, |\psi(x)|^2 = C^2 \int_{-\infty}^{\infty} dx\, x e^{-(x-x_0)^2/2x_0^2}$$

change $y = x - x_0$

$$= \underbrace{C^2 x_0 \int_{-\infty}^{\infty} dx\, e^{-y^2/2x_0^2}}_{1/C^2} + \underbrace{C^2 \int_{-\infty}^{\infty} dy\, y\, e^{-y^2/2x_0^2}}_{0 \text{ since it is an odd function}} = x_0$$

$$\Rightarrow \boxed{\langle x \rangle = x_0}$$

(d) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle (x-\langle x\rangle)^2\rangle}$

$\langle x\rangle = x_0$

$\langle (x-x_0)^2\rangle = \int_{-\infty}^{\infty} dx\,(x-x_0)^2\, C^2 e^{-(x-x_0)^2/2x_0^2}$ change to $y = x - x_0$

$\langle (x-x_0)^2\rangle = C^2 \int_{-\infty}^{\infty} dy\, y^2 e^{-y^2/2x_0^2} = \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{\infty} dy\, y^2 e^{-y^2/2x_0^2}$

use that

$\int_{-\infty}^{\infty} dy\, y^2 e^{-\lambda y^2} = -\frac{\partial}{\partial\lambda}\int dy\, e^{-\lambda y^2} = -\frac{\partial}{\partial\lambda}\sqrt{\frac{\pi}{\lambda}} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}}$

$\lambda = \frac{1}{2x_0^2} \Rightarrow \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} = \frac{1}{2}\sqrt{8x_0^6 \cdot \pi} = x_0^3\sqrt{2\pi}$

$\Rightarrow \langle (x-x_0)^2\rangle = \frac{1}{\sqrt{2\pi}\,x_0}\, x_0^3\sqrt{2\pi} = x_0^2$

$\Rightarrow$ $\boxed{\begin{array}{l}\Delta x = x_0 = 6\,\text{nm} \\ \langle x\rangle = x_0 = 6\,\text{nm}\end{array}}$