1. $\psi(x) = Axe^{-bx}$ gives $d\psi/dx = Ae^{-bx} - bAxe^{-bx}$ and $d^2\psi/dx^2 = -2bAe^{-bx} + b^2Axe^{-bx}$. Then substituting into Equation 7.2 we have

$$-\frac{\hbar^2}{2m}(-2Abe^{-bx} + b^2Axe^{-bx}) - \frac{e^2}{4\pi\varepsilon_0 x}Axe^{-bx} = EAxe^{-bx}$$

Canceling common factors gives

$$\frac{\hbar^2 b}{m} - \frac{\hbar^2 b^2}{2m}x - \frac{e^2}{4\pi\varepsilon_0} = Ex \quad \text{or} \quad \left(\frac{\hbar^2 b}{m} - \frac{e^2}{4\pi\varepsilon_0}\right) + x\left(-\frac{\hbar^2 b^2}{2m} - E\right) = 0$$

For this expression to equal zero for all *x*, both terms in parentheses must be zero. Thus

$$\frac{\hbar^2 b}{m} = \frac{e^2}{4\pi\varepsilon_0} \quad \text{or} \quad b = \frac{me^2}{4\pi\varepsilon_0\hbar^2} = \frac{1}{a_0} \qquad \text{and} \qquad E = -\frac{\hbar^2 b^2}{2m} = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}$$

2. The probability density is $P(x) = |\psi(x)|^2 = A^2x^2e^{-2bx}$. To find the maximum, we set the first derivative equal to zero:

$$\frac{dP}{dx} = 2A^2xe^{-2bx} - 2bA^2x^2e^{-2bx} = 0$$

This has solutions at $x = 0$, $x = \infty$, and $x = 1/b = a_0$. The first two give minima and the third gives the maximum.

3. The probability to find the electron in a small interval is $P(x)dx = A^2x^2e^{-2bx}dx$. Substituting the values of *A* and *b*, and evaluating the resulting expression for $x = a_0$ and $dx = 0.02a_0$ (appropriate to the interval from $x = 0.99a_0$ to $x = 1.01a_0$), we obtain

$$P(x)dx = \frac{4}{a_0^3}x^2e^{-2x/a_0}dx = \frac{4}{a_0^3}a_0^2e^{-2}(0.02a_0) = 0.0108$$