3. $$E_1 = \frac{h^2}{8mL^2} = 5.6\text{ eV}$$

With $L' = 2L$,

$$E_1' = \frac{h^2}{8mL'^2} = \frac{h^2}{8m(2L)^2} = \frac{1}{4}\frac{h^2}{8mL^2} = \frac{1}{4}(5.6\text{ eV}) = 1.4\text{ eV}$$

4. With $\lambda_n = 2L/n$,

$$\lambda_1 = \frac{2(0.144\text{ nm})}{1} = 0.288\text{ nm} \qquad \lambda_2 = \frac{2(0.144\text{ nm})}{2} = 0.144\text{ nm} \qquad \lambda_3 = \frac{2(0.144\text{ nm})}{3} = 0.096\text{ nm}$$

5. The smallest energy is (using Equation 5.3)

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\text{ eV}\cdot\text{nm})^2}{8(511{,}000\text{ eV})(0.062\text{ nm})^2} = 98\text{ eV}$$

Then $E_2 = 2^2 E_1 = 391\text{ eV}$ and $E_3 = 3^2 E_1 = 881\text{ eV}$.

6. With $L = 1.2 \times 10^{-14}$ m = 10 fm,

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\text{ MeV}\cdot\text{fm})^2}{8(940\text{ MeV})(12\text{ fm})^2} = 1.4\text{ MeV}$$

8. (a) The regions with $x < a$ and $x > a$ do not contribute to the normalization. The normalization integral is

$$\int |\psi(x)|^2 \, dx = \int_{-a}^{+a} b^2 (a^2 - x^2)^2 \, dx = b^2 \int_{-a}^{+a} (a^4 - 2a^2 x^2 + x^4) \, dx = b^2 \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{-a/2}^{+a/2}$$

Evaluating the integral and setting it equal to 1, we find

$$b^2 \left(2a^5 - \frac{4a^5}{3} + \frac{2a^5}{5} \right) = 1 \qquad \text{or} \qquad b = \sqrt{\frac{15}{16a^5}}$$

(b) $P(x)\,dx = |\psi(x)|^2 \, dx = b^2 (a^2 - x^2)^2 \, dx$, and with $x = +a/2$ and $dx = 0.010a$ we obtain

$$P(x)\,dx = \frac{15}{16a^5} \left(\frac{a^2}{4} - a^2 \right)^2 (0.010a) = 0.0053$$

(c) $P(a/2 : a) = \int_{a/2}^{a} |\psi(x)|^2 \, dx = \int_{a/2}^{a} b^2 (a^2 - x^2)^2 \, dx = \frac{15}{16a^5} \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{a/2}^{a}$

$$= \frac{15}{16a^5} \left[a^4 \left(a - \frac{a}{2} \right) - \frac{2a^2}{3} \left(a^3 - \frac{a^3}{8} \right) + \frac{1}{5} \left(a^5 - \frac{a^5}{32} \right) \right] = 0.104$$

9. With $\psi(x) = Cxe^{-bx}$, we have $d\psi/dx = Ce^{-bx} - bCxe^{-bx}$ and

$$\frac{d^2\psi}{dx^2} = -2bCe^{-bx} + b^2 Cxe^{-bx}$$

We now substitute $\psi(x)$ and $d^2\psi/dx^2$ into the Schrödinger equation:

$$-\frac{\hbar^2}{2m}(-2bCe^{-bx} + b^2Cxe^{-bx}) + U(x)Cxe^{-bx} = ECxe^{-bx}$$

Canceling the common factor of Ce^{-bx} and solving for *E*,

$$E = \frac{\hbar^2 b}{mx} - \frac{\hbar^2 b^2}{2m} + U(x)$$

The energy *E* will be a constant only if the two terms that depend on *x* cancel each other:

$$\frac{\hbar^2 b}{mx} + U(x) = 0 \qquad \text{or} \qquad U(x) = -\frac{\hbar^2 b}{mx}$$

The cancellation of the two terms depending on *x* leaves only the remaining term for the energy:

$$E = -\frac{\hbar^2 b^2}{2m}$$

11. (a) The normalization integral is

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{+\infty} A^2 x^2 e^{-2bx} dx = 2A^2 \int_0^{+\infty} x^2 e^{-2bx} dx = 2A^2 \frac{2}{(2b)^3} = 1$$

so $A = \sqrt{2b^3}$.

(b) The wave function can be written as $\psi(x) = Ae^{bx}$ for $x < 0$ and $\psi(x) = Ae^{-bx}$ for $x > 0$.

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{0} A^2 e^{2bx} dx + \int_0^{+\infty} A^2 e^{-2bx} dx = \frac{2A^2}{2b}$$

and $A = \sqrt{b}$.

12. For the $x > 0$ wave function, we have

$$\frac{d\psi}{dx} = -Abe^{-bx} \qquad \text{and} \qquad \frac{d^2\psi}{dx^2} = Ab^2 e^{-bx}$$

Substituting into the Schrödinger equation then gives

$$-\frac{\hbar^2}{2m} Ab^2 e^{-bx} + UAe^{-bx} = EAe^{-bx} \qquad \text{or} \qquad -\frac{\hbar^2 b^2}{2m} + U = E$$

This is consistent with a constant value of U, which we can take to be 0, giving $E = -\hbar^2 b^2 / 2m$. Repeating the calculation for the $x < 0$ wave function gives an identical result.

So the potential energy is a constant (zero) for $x < 0$ and for $x > 0$. What happens at $x = 0$? Note that the wave function is continuous at $x = 0$ (both give $\psi = A$ at $x = 0$) but that the derivative $d\psi/dx$ is not continuous. This suggests an infinite discontinuity in $U(x)$ at $x = 0$, and because the wave functions approach 0 as $x \to \infty$ there must be a negative potential energy that produces the bound states. So the potential energy is

$$U(x) = 0 \qquad x < 0,\ x > 0$$
$$U(x) = -\infty \qquad x = 0$$

This type of function is known as a delta function.

13. (a) $$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\ \text{eV}\cdot\text{nm})^2}{8(511{,}000\ \text{eV})(0.285\ \text{nm})^2} = 4.63\ \text{eV}$$

$4 \to 1$: $$\Delta E = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(4.63\ \text{eV}) = 69.5\ \text{eV}$$

(b) $4 \to 3$: $$\Delta E = E_4 - E_3 = 16E_1 - 9E_1 = 7E_1 = 7(4.63\ \text{eV}) = 32.4\ \text{eV}$$

$4 \to 2$: $$\Delta E = E_4 - E_2 = 16E_1 - 4E_1 = 12E_1 = 12(4.63\ \text{eV}) = 55.6\ \text{eV}$$

$3 \to 2$: $$\Delta E = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1 = 5(4.63\ \text{eV}) = 23.2\ \text{eV}$$

$3 \to 1$: $$\Delta E = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(4.63\ \text{eV}) = 37.0\ \text{eV}$$

$2 \to 1$: $$\Delta E = E_2 - E_1 = 4E_1 - E_1 = 3E_1 = 3(4.63\ \text{eV}) = 13.9\ \text{eV}$$

14. With $E_1 = 1.54\ \text{eV}$ and $E_n = n^2 E_1$ we have

$$\Delta E_3 = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(1.54\ \text{eV}) = 12.3\ \text{eV}$$

$$\Delta E_4 = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(1.54\ \text{eV}) = 23.1\ \text{eV}$$

16. (a) $P(0:L/3)=\int_0^{L/3}|\psi_1(x)|^2\,dx=\int_0^{L/3}\frac{2}{L}\sin^2\frac{\pi x}{L}\,dx=\frac{2}{\pi}\int_0^{\pi/3}\sin^2 u\,du$

$$=\frac{2}{\pi}\left(\frac{u}{4}-\frac{\sin 2u}{4}\right)\Bigg|_0^{\pi/3}=0.1955$$

(b) $P(L/3:2L/3)=\int_{L/3}^{2L/3}|\psi_1(x)|^2\,dx=\int_{L/3}^{2L/3}\frac{2}{L}\sin^2\frac{\pi x}{L}\,dx=\frac{2}{\pi}\int_{\pi/3}^{2\pi/3}\sin^2 u\,du$

$$=\frac{2}{\pi}\left(\frac{u}{4}-\frac{\sin 2u}{4}\right)\Bigg|_{\pi/3}^{2\pi/3}=0.6090$$

(c) $P(2L/3:L)=\int_{2L/3}^{L}|\psi_1(x)|^2\,dx=\int_{2L/3}^{L}\frac{2}{L}\sin^2\frac{\pi x}{L}\,dx=\frac{2}{\pi}\int_{2\pi/3}^{\pi}\sin^2 u\,du$

$$=\frac{2}{\pi}\left(\frac{u}{4}-\frac{\sin 2u}{4}\right)\Bigg|_{2\pi/3}^{\pi}=0.1955$$

17. (a) $P(x)dx=|\psi_3(x)|^2\,dx=\frac{2}{L}\sin^2\frac{3\pi x}{L}\,dx=\frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.188\text{ nm})}{0.189\text{ nm}}0.001\text{ nm}=2.63\times10^{-5}$

(b) $P(x)dx=\frac{2}{L}\sin^2\frac{3\pi x}{L}\,dx=\frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.031\text{ nm})}{0.189\text{ nm}}0.001\text{ nm}=0.0106$

(c) $P(x)dx=\frac{2}{L}\sin^2\frac{3\pi x}{L}\,dx=\frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.079\text{ nm})}{0.189\text{ nm}}0.001\text{ nm}=5.42\times10^{-3}$

(d) A classical particle has a uniform probability to be found anywhere within the region, so $P(x)\,dx=(0.001\text{ nm})/(0.189\text{ nm})=5.29\times10^{-3}$.

18. With $E=E_0(n_x^2+n_y^2)$ the levels above $50E_0$ are as follows:

n_x	n_y	E	n_x	n_y	E
6	4	$52E_0$	6	5	$61E_0$
4	6	$52E_0$	5	6	$61E_0$
7	2	$53E_0$	7	4	$65E_0$
2	7	$53E_0$	4	7	$65E_0$
7	3	$58E_0$	8	1	$65E_0$
3	7	$58E_0$	1	8	$65E_0$

The level at $E = 65E_0$ is 4-fold degenerate.

19. With $E = E_0(n_x^2 + n_y^2/4)$ the levels are as follows:

n_x	n_y	E	n_x	n_y	E
1	1	$1.25E_0$	2	3	$6.25E_0$
1	2	$2.00E_0$	1	5	$7.25E_0$
2	1	$2.25E_0$	2	4	$8.00E_0$
1	3	$3.25E_0$	3	1	$9.25E_0$
2	2	$5.00E_0$	1	6	$10.00E_0$
1	4	$5.00E_0$	3	2	$10.00E_0$

The levels at $E = 5.00E_0$ and $E = 10.00E_0$ are both 2-fold degenerate.

35. (a) $$E = n^2 E_1 = 10^2 \frac{h^2}{8mL^2} = \frac{100(hc)^2}{8mc^2L^2} = \frac{100(1240 \text{ eV} \cdot \text{nm})^2}{8(511{,}000 \text{ eV})(0.132 \text{ nm})^2} = 2160 \text{ eV}$$

(b) $$\Delta p = \sqrt{p^2} = \sqrt{2mE} = \frac{1}{c}\sqrt{2mc^2E} = \frac{1}{c}\sqrt{2(511{,}000 \text{ eV})(2160 \text{ eV})} = 4.70 \times 10^4 \text{ eV}/c$$

(c) $$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{1}{2\pi}\frac{hc}{c\Delta p} = \frac{1}{2\pi}\frac{1240 \text{ eV} \cdot \text{nm}}{4.70 \times 10^4 \text{ eV}} = 4.2 \times 10^{-3} \text{ nm}$$

37. $(x^2)_{\text{av}} = \int_0^L |\psi(x)|^2 \, x^2 dx = \frac{2}{L}\int_0^L x^2 \sin^2 \frac{n\pi x}{L} dx = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^{n\pi} u^2 \sin u \, du \quad \text{with } u = \frac{n\pi x}{L}$

The integral is a standard form that can be found in integral tables.

$$(x^2)_{\text{av}} = \frac{2L^2}{(n\pi)^3}\left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}\right]_0^{n\pi} = \frac{2L^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right] = L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

38. With $x_{\text{av}} = L/2$ from Example 5.5, we have

$$\Delta x = \sqrt{(x^2)_{\text{av}} - (x_{\text{av}})^2} = \sqrt{L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \left(\frac{L}{2}\right)^2} = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

39. (a) The particle has no preferred direction of motion, so it is equally likely to be moving in the positive and negative x directions. We therefore expect that $p_{\text{av}} = 0$.

(b) Because the potential energy is zero inside the well, the kinetic energy is equal to the total energy:

$$K = E_n \quad \text{or} \quad \frac{p^2}{2m} = \frac{h^2n^2}{8mL^2} \qquad \text{so} \qquad p^2 = \frac{h^2n^2}{4L^2}$$

For a given level n, p^2 is constant so $(p^2)_{\text{av}}$ has that same value.

(c) $$\Delta p = \sqrt{(p^2)_{\text{av}} - (p_{\text{av}})^2} = \sqrt{\frac{h^2n^2}{4L^2} - 0} = \frac{hn}{2L}$$

43. (a) With $\Delta x = \sqrt{(x^2)_{\text{av}} - (x_{\text{av}})^2}$, clearly $x_{\text{av}} = 0$ for this wave function. Then

$$(x^2)_{\text{av}} = \int_{-\infty}^{+\infty} x^2 \,|\psi(x)|^2 \, dx = 2b^{-1}\int_0^{+\infty} x^2 e^{-2x/b} dx = 2b^{-1}\frac{2}{(2/b)^3} = \frac{b^2}{2}$$

So $\Delta x = b/\sqrt{2} = 0.71b$.

(b) The maximum probability density occurs at $x = 0$, where $P(x) = |\psi(x)|^2 = b^{-1}$. We now find the location where $P(x)$ drops to half that value, that is, where $e^{-2|x|/b} = 0.5$, or $-2|x|/b = \ln(0.5)$:

$$|x| = -(b/2)\ln(0.5) \qquad \text{or} \qquad x = \pm 0.347b$$

Our estimate for Δx is then the distance between the two points where the probability is half its maximum value, so $\Delta x = 0.69b$, which agrees very well with the result of the more rigorous calculation from part (a).