

10.

$$\lambda = \frac{d \sin \phi}{2} = \frac{(0.215 \text{ nm})(\sin 55^\circ)}{2} = 0.0881 \text{ nm}$$
$$pc = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.0881 \text{ nm}} = 1.408 \times 10^4 \text{ eV}$$
$$K = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} = \frac{(1.408 \times 10^4 \text{ eV})^2}{2(0.511 \times 10^6 \text{ eV})} = 194 \text{ eV}$$

To achieve this kinetic energy, the electrons must be accelerated through a potential difference of  $\Delta V = +194 \text{ V}$ .

19. With  $\Delta v = 2.8 \times 10^4$  m/s,

$$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{\hbar}{m \Delta v} = \frac{1.05 \times 10^{-34} \text{ J} \cdot \text{s}}{(9.11 \times 10^{-31} \text{ kg})(2.8 \times 10^4 \text{ m/s})} = 4.1 \times 10^{-9} \text{ m} = 5.8 \text{ nm}$$

20. (a) 
$$\Delta p \sim \frac{\hbar}{\Delta x} = \frac{h}{2\pi \Delta x} = \frac{1}{c} \frac{hc}{2\pi \Delta x} = \frac{1}{c} \frac{1240 \text{ eV} \cdot \text{nm}}{2\pi(0.1 \text{ nm})} = 2000 \text{ eV}/c$$

(b) 
$$K = \frac{(\Delta p)^2}{2m} = \frac{(c \Delta p)^2}{2mc^2} = \frac{(2000 \text{ eV})^2}{2(0.511 \times 10^6 \text{ eV})} = 4 \text{ eV}$$

21. 
$$\Delta E \sim \frac{\hbar}{\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{2.0 \times 10^{-23} \text{ s}} = 33 \text{ MeV}$$

Measurements of the  $\Sigma^+$  rest energy are likely to fall in the range  $1385 \text{ MeV} \pm 33 \text{ MeV}$ , or from  $1352 \text{ MeV}$  to  $1418 \text{ MeV}$ .

$$22. \quad \Delta t \sim \frac{\hbar}{\Delta E} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{120 \times 10^6 \text{ eV}} = 5.5 \times 10^{-24} \text{ s}$$

25. As we did for electrons in Example 4.9, let's find the kinetic energy of an alpha particle with a momentum of  $19.7 \text{ MeV}/c$ :

$$K = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} = \frac{(19.7 \text{ MeV})^2}{2(3727 \text{ MeV})} = 0.052 \text{ MeV}$$

This is negligible compared with the typical kinetic energies of alpha particles emitted in radioactive decays. Therefore, the uncertainty principle does not limit the existence of these alpha particles inside the nucleus.

26. With  $\Delta x = 14 \text{ fm}$ , we have

$$\Delta p_x = \frac{\hbar}{\Delta x} = \frac{1}{c} \frac{\hbar c}{\Delta x} = \frac{1}{c} \frac{197 \text{ MeV} \cdot \text{fm}}{14 \text{ fm}} = 14.1 \text{ MeV}/c$$

With this uncertainty as an estimate for  $p_x$ ,

$$K = \frac{p_x^2}{2m} = \frac{c^2 p_x^2}{2mc^2} = \frac{(14.1 \text{ MeV})^2}{2(938 \text{ MeV})} = 0.11 \text{ MeV}$$

This is a very small contribution to the energy of protons or neutrons in a nucleus, which are typically 10-20 MeV.

27. 
$$y(x) = \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} A(k) \cos kx \, dk = A_0 \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} \cos kx \, dk = A_0 \left. \frac{\sin kx}{x} \right|_{k_0 - \Delta k/2}^{k_0 + \Delta k/2}$$

$$= \frac{A_0}{x} [\sin x(k_0 + \Delta k / 2) - \sin x(k_0 - \Delta k / 2)]$$

$$= \frac{A_0}{x} \left[ \sin k_0 x \cos \frac{x \Delta k}{2} + \cos k_0 x \sin \frac{x \Delta k}{2} - \left( \sin k_0 x \cos \frac{x \Delta k}{2} - \cos k_0 x \sin \frac{x \Delta k}{2} \right) \right]$$

$$= \frac{2A_0}{x} \sin \left( \frac{\Delta k}{2} x \right) \cos k_0 x$$

$$28. \quad y(x) = \int_{-\infty}^{+\infty} A(k) \cos kx \, dk = A_0 \int_{-\infty}^{+\infty} e^{-(k-k_0)^2/2(\Delta k)^2} \cos kx \, dk$$

Let  $k' = k - k_0$ .

$$y(x) = A_0 \int_{-\infty}^{+\infty} e^{-k'^2/2(\Delta k)^2} [\cos k'x \cos k_0x - \sin k'x \sin k_0x] \, dk'$$

The integral over the second term (involving the sines) vanishes because  $\sin k'x$  is an odd function of  $k'$  (the contribution of the integral from  $-\infty$  to 0 cancels the part from 0 to  $+\infty$ ). The remaining integral is

$$y(x) = 2A_0 \cos k_0x \int_0^{\infty} e^{-k'^2/2(\Delta k)^2} \cos k'x \, dk'$$

The integral is a standard form that can be found in integral tables:

$$y(x) = 2A_0 \cos k_0x \sqrt{\pi} \frac{e^{-x^2(\Delta k)^2/2}}{\sqrt{2/\Delta k}} = A_0 \Delta k \sqrt{2\pi} e^{-x^2(\Delta k)^2/2} \cos k_0x$$

$$29. \quad y(x) = A \cos(2\pi x / \lambda_1) + A \cos(2\pi x / \lambda_2) = A[\cos(2\pi x / \lambda_1) + \cos(2\pi x / \lambda_2)]$$

Using the identity  $\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$ , we get directly

$$y(x) = 2A \cos\left(\frac{\pi x}{\lambda_1} + \frac{\pi x}{\lambda_2}\right) \cos\left(\frac{\pi x}{\lambda_1} - \frac{\pi x}{\lambda_2}\right)$$

31. (a)  $v_{\text{phase}} = \omega/k$

$$v_{\text{group}} = \frac{d\omega}{dk} = \frac{d}{dk}(kv_{\text{phase}}) = v_{\text{phase}} + k \frac{dv_{\text{phase}}}{dk}$$

$$\frac{dv_{\text{phase}}}{dk} = \frac{dv_{\text{phase}}}{d\lambda} \frac{d\lambda}{dk} = \frac{dv_{\text{phase}}}{d\lambda} \frac{d}{dk} \left( \frac{2\pi}{k} \right) = \frac{dv_{\text{phase}}}{d\lambda} \left( -\frac{2\pi}{k^2} \right) = \frac{dv_{\text{phase}}}{d\lambda} \left( -\frac{\lambda}{k} \right)$$

$$v_{\text{group}} = v_{\text{phase}} - \lambda \frac{dv_{\text{phase}}}{d\lambda}$$

(b) The index of refraction  $n$  for light in glass decreases as  $\lambda$  increases (shorter wavelengths are refracted more than longer wavelengths); that is  $dn/d\lambda < 0$ . Because  $n = c/v_{\text{phase}}$ ,  $dn/d\lambda$  and  $dv_{\text{phase}}/d\lambda$  have opposite signs and so  $dv_{\text{phase}}/d\lambda > 0$ . Thus

$$v_{\text{group}} > v_{\text{phase}}.$$

32. 
$$v_{\text{phase}} = \sqrt{\frac{b}{\lambda}} = \sqrt{\frac{bk}{2\pi}} = \frac{\omega}{k} \quad \text{or} \quad \omega = \sqrt{\frac{b}{2\pi}} k^{3/2}$$

$$v_{\text{group}} = \frac{d\omega}{dk} = \sqrt{\frac{b}{2\pi}} \frac{3}{2} k^{1/2} = \frac{3}{2} \sqrt{\frac{bk}{2\pi}} = \frac{3}{2} v_{\text{phase}}$$

33. 
$$K = E - mc^2 = \sqrt{p^2 c^2 + m^2 c^4} - mc^2$$

$$\frac{dK}{dp} = \frac{1}{2} (p^2 c^2 + m^2 c^4)^{-1/2} (2pc^2) = \frac{pc^2}{\sqrt{p^2 c^2 + m^2 c^4}} = \frac{pc^2}{E} = c^2 \frac{mv / \sqrt{1 - v^2/c^2}}{mc^2 / \sqrt{1 - v^2/c^2}} = v$$

34. (a) With a node at each end (say, at  $x = 0$  and  $x = L$ ) and no other nodes, we must have one half-wave between the two nodes. Thus  $L = \lambda_1/2$  or  $\lambda_1 = 2L$ . If there is an additional node at the midpoint ( $x = L/2$ ), then there is a full wave between the two ends, and  $L = \lambda_2$  or  $\lambda_2 = 2L/2$ . The next shorter wavelength has (in addition to the nodes at either end) nodes at  $x = L/3$  and  $x = 2L/3$ , so there are three half-waves between the ends:  $L = 3\lambda_3/2$  or  $\lambda_3 = 2L/3$ . Continuing in this way, we see that in the  $n^{\text{th}}$  case there are  $n$  half-waves in the length  $L$ , so  $L = n(\lambda_n/2)$  or  $\lambda_n = 2L/n$ .

(b) With  $p_n = h/\lambda_n = nh/2L$ , we see that  $cp_n$  is of order keV, so nonrelativistic equations can safely be used:



$$K_n = \frac{p_n^2}{2m} = \frac{c^2 p_n^2}{2mc^2} = \frac{n^2 h^2 c^2}{8mc^2 L^2} = n^2 \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511,000 \text{ eV})(0.50 \text{ nm})^2} = n^2 (1.50 \text{ eV})$$

Thus  $K_1 = 1.50 \text{ eV}$ ,  $K_2 = 6.00 \text{ eV}$ ,  $K_3 = 13.5 \text{ eV}$ .

36. (a) The mass of a nitrogen molecule is 14 u. The average molecular kinetic energy is  $\frac{3}{2}kT$ , so the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2 K}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2(14 \text{ u})(931.5 \times 10^6 \text{ eV/u})(1.5)(8.617 \times 10^{-5} \text{ eV/K})(293 \text{ K})}} = 0.0279 \text{ nm}$$

- (b) The number of nitrogen molecules per unit volume is

$$n = \frac{\rho N_A}{M} = \frac{(1.292 \text{ kg/m}^3)(6.02 \times 10^{23} \text{ molecules/mole})}{(0.028 \text{ kg/mole})} = 2.78 \times 10^{25} \text{ molecules/m}^3$$

and the average spacing between molecules is  $n^{-1/3} = 3.30 \times 10^{-9} \text{ m} = 3.3 \text{ nm}$ . The de Broglie wavelength is 2 orders of magnitude smaller than the molecular spacing, so that quantum effects are unimportant in gases at room temperature.

- (c) Let's estimate that quantum effects would be significant if the de Broglie wavelength were about 1/10 of the molecular separation (0.33 nm):

$$p = \frac{h}{\lambda} = \frac{1}{c} \frac{hc}{\lambda} = \frac{1}{c} \frac{1240 \text{ eV} \cdot \text{nm}}{0.33 \text{ nm}} = 3760 \text{ eV}/c$$

$$K = \frac{p^2}{2m} = \frac{p^2 c^2}{2mc^2} = \frac{(3760 \text{ eV})^2}{2(28 \text{ u})(931.5 \times 10^6 \text{ eV/u})} = 2.71 \times 10^{-4} \text{ eV}$$

The molecules have this tiny amount of average kinetic energy at a temperature

$$T = \frac{2K}{3k} = \frac{2(2.71 \times 10^{-4} \text{ eV})}{3(8.617 \times 10^{-5} \text{ eV/K})} = 2.1 \text{ K}$$

Nitrogen is no longer a gas at this temperature, so our calculation using the formula for the mean molecular energy of gases is not correct. However, it does suggest that if quantum effects are to become important in gases, they will occur only at low temperatures. (Recall the discussion in Chapter 1 about how the equipartition of energy fails for the rotational and vibrational motions of some gases at even moderate temperatures, so other effects of quantum behavior may be observable at these temperatures.)

43. From Eq. 4.15 with  $p_{x,\text{av}} = 0$ , we have  $(\Delta p_x)^2 = (p_x^2)_{\text{av}}$  and similarly in the  $y$  direction we have  $(\Delta p_y)^2 = (p_y^2)_{\text{av}}$ . The kinetic energy is  $K = p^2/2m = (p_x^2 + p_y^2)/2m$ , and using the minimum estimates for the uncertainties we have

$$K = \frac{(\Delta p_x)^2 + (\Delta p_y)^2}{2m} = \frac{(\hbar / 2\Delta x)^2 + (\hbar / 2\Delta y)^2}{2m} = \frac{\hbar^2 c^2}{8mc^2} \left[ \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right]$$

$$= \frac{(197 \text{ eV} \cdot \text{nm})^2}{8(511,000 \text{ eV})} \left[ \frac{1}{(1.25 \text{ nm})^2} + \frac{1}{(2.76 \text{ eV})^2} \right] = 7.3 \text{ meV}$$