Physics 2D — Quiz 5 Solutions — Fall 2023

Problem 1

Energy levels are $E_n = -\frac{E_0}{n^2}$, $E_0 = 13.6\,eV$

Transitions from the ground state to n: photon energy is

$$\frac{hc}{\lambda_{1n}} = E_0\left(1 - \frac{1}{n^2}\right) \Rightarrow \lambda_{1n} = \frac{hc}{E_0\left(1 - \frac{1}{n^2}\right)}$$

$$\lambda_{1n} = \frac{1240}{13.6\left(1 - \frac{1}{n^2}\right)}\,nm = \frac{91.176}{(1 - 1/n^2)}$$

$\Rightarrow$ $\lambda_{12} = 121.57\,nm$, $\lambda_{13} = 102.57\,nm$, $\lambda_{14} = 97.25\,nm$

so incident radiation in range $100\,nm < \lambda < 200\,nm$ can excite electrons from ground state to $n=2$ and $n=3$ only.

Subsequent emissions will be:

$n = 2$ to $n=1$. $\boxed{\lambda_{12} = 121.57\,nm}$

$n = 3$ to $n=1$. $\boxed{\lambda_{13} = 102.57\,nm}$

$n = 3$ to $n = 2$: $\lambda_{23} = \frac{91.176}{\frac{1}{4} - \frac{1}{9}} = \boxed{656.47\,nm}$

Problem 2

$$L = m_e v r = n\hbar \quad . \quad \text{We have } r = a_0,\ L = 3\hbar \Rightarrow$$

$$m_e v a_0 = 3\hbar \Rightarrow \quad v = \frac{3\hbar}{m_e a_0}$$

$$\frac{v}{c} = \frac{3\hbar c}{m_e c^2 a_0} = \frac{3 \times 197.3\ \text{eV nm}}{511{,}000\ \text{eV} \times 0.0529\ \text{nm}} = 0.0219$$

$$\boxed{\frac{v}{c} = 0.0219} \ (a)$$

(b) It's non-relativistic

$$K = \frac{1}{2} m_e v^2 = \frac{1}{2} m_e c^2 \left(\frac{v}{c}\right)^2 = \boxed{122.5\,\text{eV}}$$

(c) The total energy is $E = K + U \Rightarrow U = -K + E$

The state of the electron is $n = 3$, since $L = n\hbar$

The radius is $$r_n = \frac{a_0 n^2}{Z} = a_0 \Rightarrow Z = n^2 = 9$$

The energy is $$E_n = -E_0 \frac{Z^2}{n^2} = -E_0 \cdot \frac{9^2}{3^2} = -9E_0 = -122.4\,\text{eV}$$

There is some rounding error, since $E_n = -K_n$ for Bohr atom

$$U = -K + E = -122.5\,\text{eV} - 122.4\,\text{eV} = \boxed{-244.9\,\text{eV} = U}$$

Problem 3

$$L^2_{op} = -\hbar^2\left[\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad ; \quad [L_z]_{op} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\psi(r,\theta,\phi) = R(r)\,(3\cos^2\theta - 1) \equiv R(r)\,Y(\theta,\phi)$$

(a) z-component of angular momentum is

$$L_z = m_\ell \hbar$$

and $[L_z]_{op}\,Y = L_z Y = m_\ell \hbar\, Y$

with $Y(\theta,\phi) = 3\cos^2\theta - 1$

since $\frac{\partial}{\partial\phi} Y = 0 \Rightarrow \boxed{m_\ell = 0} \Rightarrow \boxed{L_z = 0}$

(b) $L^2_{op}\, Y = \hbar^2 \ell(\ell+1)\, Y$

$$\frac{\partial}{\partial\theta}(3\cos^2\theta - 1) = -6\cos\theta\sin\theta$$

$$\frac{\partial^2}{\partial\theta^2}(3\cos^2\theta - 1) = -\frac{\partial}{\partial\theta}(6\cos\theta\sin\theta) = 6\sin^2\theta - 6\cos^2\theta$$

$$\Rightarrow L^2_{op}\, Y = -\hbar^2\left[\frac{\cos\theta}{\cancel{\sin\theta}}\cdot(-6\cos\theta\cancel{\sin\theta}) + 6\sin^2\theta - 6\cos^2\theta + 0\right] =$$

$$= -\hbar^2\left[-6\cos^2\theta + 6\sin^2\theta - 6\cos^2\theta\right] = -6\hbar^2(\sin^2\theta - 2\cos^2\theta) =$$

~~and~~ $\cancel{L^2_{op}\psi}$ $= -6\hbar^2(1 - \cos^2\theta - 2\cos^2\theta) = 6\hbar^2(3\cos^2\theta - 1)$

and $\hbar^2\ell(\ell+1)\,Y = \hbar^2\ell(\ell+1)(3\cos^2\theta - 1)$

$$\Rightarrow \ell(\ell+1) = 6 \Rightarrow \boxed{\ell = 2}$$

(c) Always true that $\ell \le n-1 \Rightarrow n \ge \ell + 1 \Rightarrow \boxed{n = 3, 4, 5, \ldots}$

Problem 4

$$R(r) = C r^3 e^{-r/2a_0}$$

$$P(r) = r^2 R(r)^2 = C^2 r^8 e^{-r/a_0}$$

(a) Most probable radius: r_m

$$P'(r_m) = 0 = 8 r_m^7 - \frac{r_m^8}{a_0} \Rightarrow \boxed{r_m = 8 a_0}$$

(b) $U(r) = -\dfrac{ke^2 Z}{r}$; $\langle U(r) \rangle = -kZe^2 \langle \frac{1}{r} \rangle$

$$\langle \tfrac{1}{r} \rangle = \frac{\int_0^\infty dr \, \frac{1}{r} P(r)}{\int_0^\infty dr \, P(r)} = \frac{\int_0^\infty dr \, r^7 e^{-r/a_0}}{\int_0^\infty dr \, r^8 e^{-r/a_0}}$$; use $\int_0^\infty dr \, r^n e^{-\lambda r} = \dfrac{n!}{\lambda^{n+1}}$

$$= \frac{7!}{(1/a_0)^{\cancel{7}} \cdot 8!} (1/a_0)^8 = \frac{1}{8a_0}$$

$$\Rightarrow \langle U \rangle = -\frac{kZe^2}{8a_0} = -Z \cdot \frac{1.44 \text{ eV nm}}{8 \times 0.0529 \text{ nm}}$$

$$\Rightarrow \boxed{\langle U \rangle = -3.40 Z \text{ eV}}$$

(c) Assuming r_m is the Bohr radius for this orbit:

$$r_m = \frac{a_0}{Z} n^2 = 8a_0 \Rightarrow Z = n^2/8$$

The exponential term in the radial wavefunction is e^{-Zr/na_0}

$$\Rightarrow \frac{Z}{n} = \frac{1}{2} \Rightarrow Z = \frac{n}{2} = \frac{n^2}{8} \Rightarrow n = 4 \Rightarrow \boxed{Z = 2}$$

Problem 5

electrons are initially in the $n=1$, $l=0$ state of hydrogen. $m_l=0$. Their spin quantum number is $m_s = \pm \frac{1}{2}$. In the absence of a magnetic field, they have the same energy, so there is just one energy value for that state, $E_1 = -E_0 = -13.5984\ eV$. That is the ionization energy.

When a magnetic field B is turned on, the states with $m_s = \frac{1}{2}$ and $m_s = -\frac{1}{2}$ have different energy:

$$E_1(m_s) = E_1 - \vec{\mu}_s \cdot \vec{B} = E_1 + \frac{e}{2m_e} g S_z = E_1 + \frac{e\hbar}{2m_e} \cdot g \cdot m_s$$

$$\Rightarrow E_1(m_s) = E_1 \pm \frac{e\hbar}{2m_e} B = E_1 \pm \mu_B B, \quad \text{since } g=2,\ \mu_B = \frac{e\hbar}{2m_e}$$

The ionization energy is $I(m_s) = -E_1(m_s)$

The difference in ionization energies with and without B is $0.001\,eV$

$$\Rightarrow \mu_B B = 0.001\,eV \Rightarrow B = \frac{0.001}{5.79 \times 10^{-5}}\ T$$

$$\Rightarrow \boxed{B = 17.27\,T}$$

The energy is lower for $m_s = -\frac{1}{2}$, higher for $m_s = +\frac{1}{2}$, with $\vec{B} = B\hat{z}$

So the atoms with lower energy have higher ionization energy.

So the ionization energy is $13.5994\,eV$ for the atoms with $m_s = -\frac{1}{2}$, and $13.5974\,eV$ for the atoms with $m_s = +\frac{1}{2}$.

Because at room temperature $k_B T = \frac{300}{11{,}604}\ eV = 0.026\,eV \gg 0.001\,eV$, there will be about half of the atoms with $m_s = +1/2$, half with $m_s = -1/2$

$$\frac{N(m_s = 1/2)}{N(m_s = -\frac{1}{2})} = e^{-\Delta E / k_B T} = e^{-0.002/0.026} = e^{-0.077} = 0.93$$