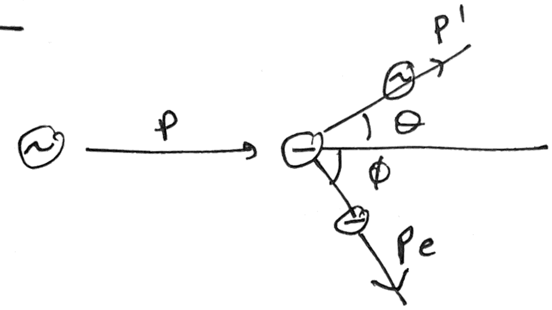
Problem 1



$p = p_e = 100{,}000\ eV/c$. $p = \frac{h}{\lambda} \Rightarrow \lambda = \frac{hc}{pc} = \frac{1240\ eV\,nm}{100{,}000\ eV} = 0.0124\,nm$

$\frac{hc}{\lambda} - \frac{hc}{\lambda'} = K_e$ kinetic energy of electron

both relativistic or non-relativistic formulas can be used. Let's use relativistic

$E = \sqrt{p_e^2c^2 + m_e^2c^4}$, $K = E - m_ec^2 = 9693\,eV$

$\frac{hc}{\lambda'} = \frac{hc}{\lambda} - K_e = 100{,}000\,eV - 9693\,eV = 90{,}307\,eV = p'c$

$\lambda' = \frac{hc}{p'c} = \frac{1240\,eV\,nm}{90{,}307\,eV} = 0.01373\,nm$

So: $\lambda = 0.0124\,nm$, $\lambda' = 0.01373\,nm$

$\lambda' - \lambda = \lambda_c(1-\cos\theta) \Rightarrow 1 - \cos\theta = \frac{0.01373 - 0.0124}{0.00243} = 0.547$

$\Rightarrow \boxed{\theta = 63.08^\circ}$ (a)

(b) momentum in y direction: $p_e \sin\phi = p'\sin\theta \Rightarrow$

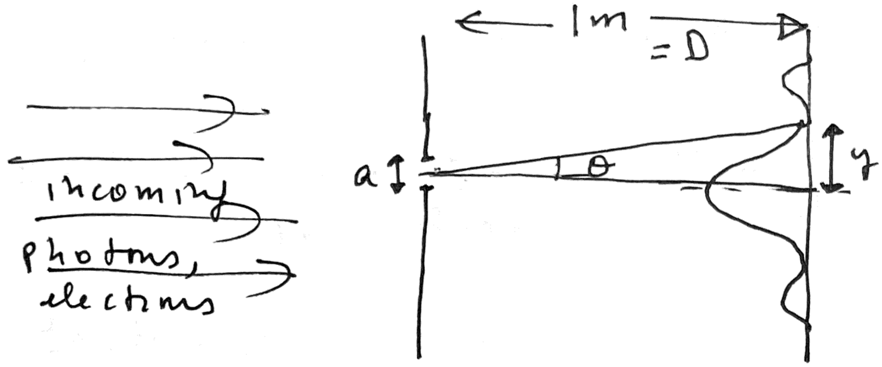
$\sin\phi = \frac{p'}{p_e}\sin\theta = \frac{90{,}307}{100{,}000} \cdot \sin(63^\circ) = 0.8046 \Rightarrow$

$\boxed{\phi = 53.58^\circ}$

Check: we should have $p = p_e\cos\phi + p'\cos\theta$

$100{,}000 = 100{,}000 \cdot \cos(53.58^\circ) + 90{,}307 \cdot \cos(63.08^\circ) =$
$= 59{,}370 + 40{,}879 = 100{,}249$ ok to 3 digits.

Problem 2



Any wave going through a narrow slit gives rise to a diffraction pattern, with the first minimum at angle θ satisfying

$$a \sin\theta = \lambda$$

For the photons, $E = hf = \frac{hc}{\lambda} = 10\,eV \Rightarrow$

$$\lambda = \frac{hc}{10\,eV} = \frac{1240\,eV\,nm}{10\,eV} = 124\,nm$$

$$a = 0.001 \cdot 10^{-3}\,m = 10^{-6}\,m = 10^{3}\,nm$$

$$\Rightarrow \sin\theta = \frac{\lambda}{a} = \frac{124\,nm}{1000\,nm} = 0.124 \Rightarrow \boxed{\theta = 7.12^\circ}$$

and $\tan\theta = \frac{y}{D} \Rightarrow y = D\tan\theta \sim D\sin\theta = 1m \times 0.124 = 0.124\,m$

$\Rightarrow \boxed{\text{distance} = 124\,mm}$ for photons

For the electrons: $K = 10\,eV$, non-relativistic

$$p = \sqrt{2m_e K} \Rightarrow pc = \sqrt{2m_e c^2 K} = 3197\,eV$$

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240\,eV\,nm}{3197\,eV} = 0.39\,nm$$

$$\sin\theta = \frac{\lambda}{a} = \frac{0.39\,nm}{1000\,nm} = 0.00039$$

$$y = D\sin\theta = 10^{3}\,mm \times 0.00039 = 0.39\,mm$$

$\boxed{y = \text{distance} = 0.39\,mm}$ for electrons

Alternative solution: use uncertainty principle

$\Delta y \, \Delta p_y \sim \hbar \Rightarrow \Delta p_y \sim \frac{\hbar}{a}$ = momentum in y direction

$$\tan\theta = \frac{\Delta p_y}{p_x} = \frac{\hbar}{a \cdot p_x}$$

$p_x = \frac{h}{\lambda}$, with λ = de Broglie wavelength

$$\tan\theta = \frac{\hbar}{a \cdot h} \cdot \lambda = \frac{\lambda}{2\pi a}$$

Answers are as given in previous page, divided by 2π.

This is just an estimate, both answers are valid.

Problem 3

$$\Psi(x,t) = \int_{k_1}^{k_2} dk\, a(k)\, e^{i(kx - \omega(k)t)}$$

$$a(k) = A\left(k_3^2 - 7(k-k_0)^2\right)$$

(a) The spread in k is $\Delta k = k_2 - k_1 = 20\,nm^{-1}$

Using $\Delta x \Delta k \sim 1 \Rightarrow \Delta x \sim \frac{1}{\Delta k} = \frac{1}{20\,nm^{-1}} = 0.05\,nm$

Position is uncertain with uncertainty $\boxed{\Delta x = 0.05\,nm}$

(b) $\Delta p = \hbar \Delta k = \frac{\hbar c \Delta k}{c} = \frac{197.3\,eV\,nm \cdot 20\,nm^{-1}}{c} = 3946\,\frac{eV}{c}$

$$\boxed{\Delta p = 3946\,\frac{eV}{c}}$$

(c) Group velocity: $\omega = \frac{\hbar k^2}{2m}$, $v_g = \left.\frac{\partial \omega}{\partial k}\right|_{k_0} = \frac{\hbar k_0}{m}$

evaluated at center of wavepacket.

$$v_g = \frac{\hbar c k_0}{mc^2} c = \frac{197.3 \times 150}{511{,}000} c = 0.0579c = 1{,}7{,}374{,}755\,m/s$$

$$\boxed{v_g = 1.74 \times 10^7\,m/s}$$

phase velocity $= \left.\frac{\omega}{k}\right|_{k_0} = \frac{\hbar k_0}{2m} = \frac{v_g}{2}$

$$\Rightarrow \boxed{v_p = 0.87 \times 10^7\,m/s}$$

Problem 4

$$m_e = 0.510998950\vdots00(15)\ \text{MeV}/c^2$$

the uncertainty in the rest energy $m_e c^2 = E$ is

$$\Delta E \cong 0.00000000015\ \cancel{M}eV = 1.5\times10^{-10}\,MeV = 1.5\times10^{-4}\,eV$$

Uncertainty principle $\Delta t\, \Delta E \sim \hbar$ gives the lifetime $\tau = \Delta t$

$$\tau \sim \frac{\hbar}{\Delta E} = \frac{\hbar c}{\Delta E c} = \frac{197.3\ \cancel{eV}\cancel{nm}\cdot s}{1.5\times10^{-4}\,\cancel{eV}\cdot 3\times10^{8}\cdot 10^{9}\,\cancel{nm}} = 4.38\times10^{-12}\,s.$$

So from the fact that we know the mass of the electron to this accuracy, we can infer that its lifetime $\boxed{\tau > 4.38\cdot10^{-12}\,s}$ (a)

(b) On the other hand, we know that the electron's lifetime is much larger than 10^{-12} s, since we don't see electrons decaying. Therefore Yes, as experiments get more accurate we will know the mass of the electron to more decimal places without violating the uncertainty principle.

Problem 5

$L = 0.001\,\text{nm}$

$\lambda = 2L$ largest possible wavelength $\quad \lambda = 0.002\,\text{nm}$

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{1240\,\text{eV nm}}{0.002\,\text{nm}\cdot c} = 620{,}000\,\text{eV}/c$$

this electron is relativistic, since $pc > m_e c^2$

$$E = \sqrt{p^2c^2 + m_e^2c^4} = \gamma m_e c^2 \Rightarrow$$

$$\gamma = \frac{\sqrt{p^2c^2 + m_e^2c^4}}{m_e c^2} = \sqrt{\frac{p^2c^2}{(m_ec^2)^2} + 1} = 1.572$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \Rightarrow$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = 0.772$$