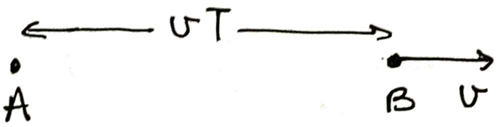
Physics 2D   Final Exam Solutions   Fall 2023

Problem 1

$$v/c = \frac{3}{5}, \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{5}{4}$$

let $T = 1$ year



In lab frame. At T, B is at distance vT. A lights up candle, light travels at speed c, reaches B in time Δt, with

$$\Delta t\, c = vT + v\Delta t \Rightarrow \Delta t(c-v) = vT \Rightarrow \Delta t = \frac{v}{c-v} T$$

So total time in lab frame is:

$$t = T + \Delta t = T\left(1 + \frac{v}{c-v}\right) = \frac{Tc}{c-v} = T\frac{1}{1-v/c}$$

In B's reference frame the time that passed is less, it's the proper time

$$t_0 = \frac{t}{\gamma} = T\frac{1}{1-\frac{v}{c}}\sqrt{1-\frac{v^2}{c^2}} = T\sqrt{\frac{1+v/c}{1-v/c}} = T\sqrt{\frac{8\cdot 5}{5\cdot 2}} = \boxed{2T}$$

so as measured in B's frame, B is 22 years old when light reaches him/her

(b) The situation is symmetric, so A is 22 years old when the light from B reaches her/him.

Alternative solution: do (b) first:

when B lights candle, in the lab frame the time that passed is $\gamma T = \frac{5}{4}T$

So B is at distance $d = \frac{5}{4}vT$. Light travels distance d in time

$t_{light} = \frac{d}{c} = \frac{5}{4}\frac{v}{c}T$. Total time is $t = \frac{5}{4}T\left(1+\frac{v}{c}\right) = \frac{5}{4}T\left(1+\frac{3}{5}\right) = \boxed{2T}$

Problem 2

$\lambda_m T = 2.8978 \cdot 10^{-3}\ \text{m K}$. $\lambda_m = 1000\ \text{nm} = 1000 \times 10^{-9}\ \text{m} = 10^{-6}\ \text{m}$

$$T = \frac{2.8978 \cdot 10^{-3}}{10^{-6}}\ \text{K} \Rightarrow \boxed{T = 2898\ \text{K}}$$

(b) The temperature dependence of the power emitted is given by

$$P = C \cdot \frac{1}{e^{hc/\lambda k_B T} - 1}$$ where C is independent of T

We have: $\dfrac{hc}{\lambda_m k_B T} = 4.96$. For $\lambda = 100{,}000\ \text{nm} = 100\ \lambda_m$

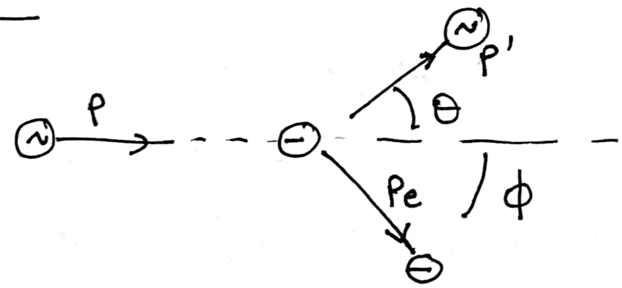
$$\frac{hc}{\lambda k_B T} = 0.0496 \ll 1$$, use $e^x \sim 1 + x$ if $x \ll 1$, so

$$P = C \frac{1}{1 + \frac{hc}{\lambda k_B T} - 1} = C' \cdot T$$

So $$\frac{P(T = 2998\ \text{K})}{P(T = 2898\ \text{K})} = \frac{2998}{2898} = 1.0345$$

So if $P(T = 2898\ \text{K}) = 100\ \text{W}$, $\boxed{P(2998\ \text{K}) = 103.45\ \text{W}.}$

Problem 3



Compton: $\lambda' = \lambda + \lambda_c(1-\cos\theta)$. $\lambda_c = \dfrac{h}{m_e c} = 0.00243\,\text{nm}$

We have: $\theta = 60° \Rightarrow \cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$$\lambda' = \lambda_c = \frac{h}{m_e c}$$

$$\Rightarrow \lambda_c = \lambda + \lambda_c \cdot \frac{1}{2} \Rightarrow \lambda = \frac{\lambda_c}{2} = \frac{h}{2m_e c}$$

$$p = \frac{h}{\lambda} = 2m_e c, \quad p' = \frac{h}{\lambda'} = m_e c$$

Energy conservation: K_e = electron kinetic energy, pc = photon energy

$$K_e = pc - p'c = 2m_e c^2 - m_e c^2 = m_e c^2$$

$$\Rightarrow \boxed{K_e = 511{,}000\,\text{eV}}$$

(b) Momentum conservation:

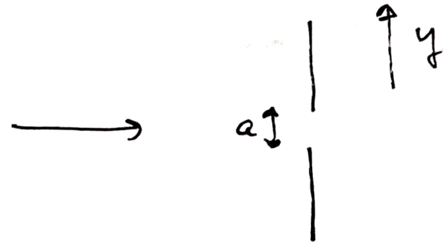
$$p = p'\cos\theta + p_e\cos\phi \Rightarrow p_e\cos\phi = p - p'\cos\theta$$

$$p'\sin\theta = p_e\sin\phi$$

$$\Rightarrow \tan\phi = \frac{p'\sin\theta}{p - p'\cos\theta} = \frac{m_e c \cdot \frac{\sqrt{3}}{2}}{2m_e c - m_e c \cdot \frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{1}{\sqrt{3}}$$

$$\tan\phi = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\phi = 30°}$$

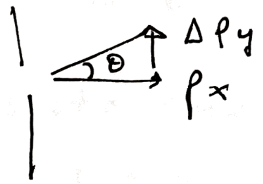
Problem 4



according to uncertainty principle, $\Delta y \, \Delta p_y \sim \hbar$

when photons/electrons go through slit, $\Delta y = a \Rightarrow$

$\Delta p_y \sim \frac{\hbar}{a}$. If p_x = momentum in x direction



$$\tan\theta = \frac{\Delta p_y}{p_x} = \frac{\hbar}{p_x a}$$

For photons, $E = pc \Rightarrow \quad p_x = \frac{E}{c} = \frac{2\text{ eV}}{c} \Rightarrow$

$$\tan\theta = \frac{\hbar c}{2\text{eV} \cdot a} = \frac{197.3\text{ eV nm}}{2\text{eV} \cdot 0.5 \times 10^3 \text{ nm}} = 0.197$$

$$\Rightarrow \boxed{\theta = 0.195 \text{ rad} = 11.2^\circ}$$

For electrons : nonrelativistic

$$p_x = \sqrt{2 K m_e} \Rightarrow p_x c = \sqrt{2 \cdot 2 \cdot 511{,}000} \text{ eV} = 1430 \text{ eV}$$

$$\Rightarrow \tan\theta = \frac{197.3 \text{ eV nm}}{1430\text{eV} \cdot 500\text{nm}} = 0.00028$$

$$\Rightarrow \boxed{\theta = 0.016^\circ}$$

Problem 5

For H atoms in ground state, the longest wavelength photons they can absorb is $n=1 \to n=2$,

$$\frac{hc}{\lambda} = E_0\left(1-\frac{1}{4}\right) = \frac{3}{4}E_0 \Rightarrow \lambda = \frac{hc}{E_0}\cdot\frac{4}{3}$$

$E_0 = 13.6\,eV$, $hc = 1240\,eV\,nm$, $\lambda = 121.57\,nm$

For atoms moving towards the photons, they will see a slightly <u>smaller</u> λ due to the Doppler shift. For atoms moving away from the photons, they see a slightly <u>larger</u> λ.

So the smallest wavelength of the radiation should be <u>slightly larger</u> than 121.57 nm, so that when Doppler-shifted down it will cause absorption, and when Doppler-shifted up it won't.

(b) Atoms <u>slow down</u>

$$p' = m_H v' = m_H v - \frac{h}{\lambda}$$

$$v' = v - \frac{h}{m_H\lambda} = v - \frac{hc}{m_H c\lambda} = v - \frac{3}{4}\frac{E_0}{m_H c^2}c \equiv v - \Delta v$$

$$m_H c^2 = 938\,MeV$$

$$\Delta v = \frac{3}{4}\cdot\frac{13.6\,eV}{938\cdot10^6\,eV}\cdot 3\cdot10^8\,\frac{m}{s} = \boxed{3.3\,\frac{m}{s}}$$

$$\boxed{\text{So atoms slow down from } 1000\,\frac{m}{s} \text{ to } 996.7\,m/s}$$

Problem 6

$$E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m_eL^2}(n_1^2+n_2^2)$$

(n_1, n_2)		E_{n_1,n_2}
$(3,1);(1,3)$	——	$10\,E_0$
$(2,2)$	——	$8\,E_0$
$(1,2);(2,1)$	$\uparrow\downarrow\uparrow$	$5\,E_0$
$(1,1)$	$\uparrow\downarrow$	$2E_0$

$L = 0.56\,\text{nm}$, $\frac{\hbar^2\pi^2}{2m_e L^2} = 1.20\,\text{eV} \equiv E_0$

$\frac{\hbar^2}{2m_e} = 0.0381\,\text{eVnm}^2$

So: $E_{n_1,n_2} = E_0(n_1^2+n_2^2)$

We have 5 electrons.

2 electrons of opposite spin per state.

Total energy:

$$\boxed{E = 2\times(2E_0) + 3\times(5E_0) = 19\,E_0 = 22.8\,\text{eV}}$$

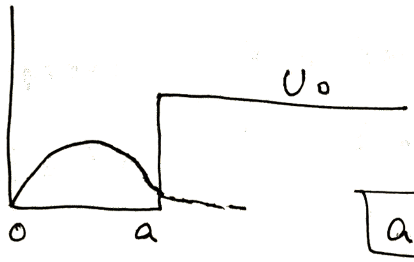
(b) Longest wavelength photon: either $E_{1,1} \to E_{1,2}$ or $E_{1,2} \to E_{2,2}$ give same

$$\frac{hc}{\lambda} = 3\,E_0 \Rightarrow \lambda = \frac{hc}{3E_0} = \frac{1240\,\text{nm}}{3.6} \Rightarrow \boxed{\lambda = 344.44\,\text{nm}}$$

(c) Next longest $E_{1,2} \to E_{3,1}$: $\Delta E = 5\,E_0$

$$\boxed{\lambda = \frac{hc}{5E_0} = 206.67\,\text{nm}}$$

Problem 7



$\boxed{a = 0.5\,\text{nm}}$

If $U_0 = \infty$, it is an infinite well: ground state is

$$E_1 = \frac{\hbar^2 k_1^2}{2m_e} = \frac{\hbar^2 \pi^2}{2m_e a^2} = \boxed{1.504\,eV}$$

$$\Psi_1(x) = C \cdot \sin(kx) \quad . \quad \boxed{k = \frac{\pi}{a} = 6.283\,\text{nm}^{-1}}$$

(b) For $U_0 \neq \infty$ we have: from Schr eq $-\frac{\hbar^2}{2m_e}\frac{d^2\Psi}{dx^2} + U(x)\Psi = E\Psi$

$0 < x < a$: $\Psi(x) = C\sin(kx)$; $k = \sqrt{\frac{2m_e E}{\hbar^2}}$

$x > a$: $\Psi(x) = De^{-\alpha x}$

$$\alpha = \sqrt{\frac{2m_e(U_0 - E)}{\hbar^2}} \Rightarrow U_0 = E + \frac{\hbar^2\alpha^2}{2m_e}$$

Continuity: $C\sin(ka) = De^{-\alpha a}$

Cont. of Ψ': $kC\cos(ka) = -\alpha De^{-\alpha a}$

Taking ratio: $\frac{1}{k}\tan(ka) = -\frac{1}{\alpha} \Rightarrow \alpha = -\frac{k}{\tan(ka)}$

$$E = 1\,eV \Rightarrow k = \sqrt{\frac{2mE}{\hbar^2}} = 5.123\,\text{nm}^{-1}$$

$$\alpha = \frac{-5.123 \quad \text{nm}^{-1}}{\tan(5.123 \times 0.5)} = \cancel{14.5\ldots}\ 7.818\,\text{nm}^{-1}$$

$$U_0 = E + 0.0381 \cdot 7.818^2\,eV = E + 2.33\,eV$$

$$\Rightarrow \boxed{U_0 = 3.33\,eV}$$

Problem 8

Distance of closest approach: K_α = kinetic energy of α particle

$$U = \frac{ke^2 \cdot 2Z}{d} = K_\alpha$$

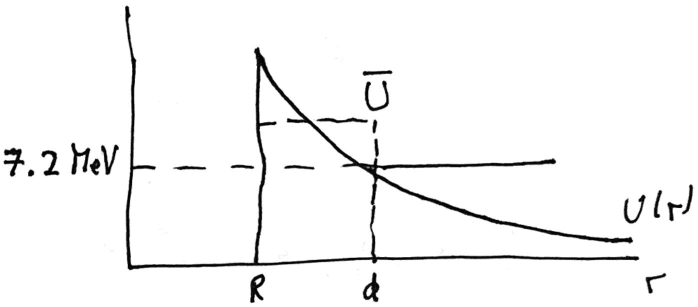
$R = 0.666 \times 10^{-5}$ nm

$$d = \frac{ke^2 \cdot 2Z}{K_\alpha} = \frac{1.44\,\text{eV nm} \cdot 2 \cdot 50}{7.2 \times 10^6\,\text{eV}} = 2 \times 10^{-5}\,\text{nm}$$

So: $U(r=d) = 7.2$ MeV

$U(r=R) = 21.6$ MeV

On average, $\bar{U} = 14.4$ MeV



Tunneling transmission coef. T gives probability that incoming α particle will tunnel through barrier and get inside the nucleus.

$$T = e^{-\sqrt{\frac{2m_\alpha}{\hbar^2}(\bar{U}-E)}\cdot(d-R)}$$

Mass of α particle: $m_\alpha = 2(938.26 + 939.55)\,\frac{\text{MeV}}{c^2} = 3756\,\frac{\text{MeV}}{c^2}$

$$\frac{2m_\alpha}{\hbar^2}(\bar{U}-E) = \frac{2 \times 3756 \times 10^6\,\text{eV} \cdot 7.2 \times 10^6\,\text{eV}}{197.3^2\,\text{eV}^2\text{nm}^2} = 1.389 \times 10^{12}\,\text{nm}^{-2}$$

$$\Rightarrow \sqrt{\frac{2m_\alpha}{\hbar^2}(\bar{U}-E)}\,(d-R) = 1.179 \times 10^6\,\text{nm}^{-1} \cdot (2 - 0.6666) \cdot 10^{-5}\,\text{nm} = 15.7$$

$$\Rightarrow \boxed{T = e^{-15.7} = 1.49 \times 10^{-7}}$$

For every 10^8 incident α-particles, $\boxed{\text{15 will get into the nucleus.}}$

Problem 9

$$U(x) = -\frac{ke^2}{x}, \quad \psi(x) = Cxe^{-x/a_0}, \quad a_0 = \frac{\hbar^2}{m_e k e^2}$$

(a) Calculate $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$$\rho(x) = |\psi(x)|^2 = C^2 x^2 e^{-2x/a_0}$$

$$\langle x\rangle = \frac{\int_0^\infty dx\, x^3 e^{-2x/a_0}}{\int_0^\infty dx\, x^2 e^{-2x/a_0}} = \frac{3!\, \left(\frac{2}{a_0}\right)^2}{\left(\frac{2}{a_0}\right)^3 \cdot 2!} = \frac{3}{2} a_0$$

$$\langle x^2\rangle = \frac{4!\, \left(\frac{2}{a_0}\right)^3}{\left(\frac{2}{a_0}\right)^4 \cdot 3!} = 3a_0^2$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{3 - \frac{9}{4}}\, a_0 = \boxed{\frac{\sqrt{3}}{2} a_0}$$

(b)

Energy $E = -\frac{ke^2}{2a_0} = U + K = \langle U\rangle + \frac{\langle p^2\rangle}{2m_e}$

$$\left\langle \frac{1}{x}\right\rangle = \frac{\int_0^\infty dx\, x e^{-2x/a_0}}{\int dx\, x^2 e^{-2x/a_0}} = \frac{1!\, \left(\frac{2}{a_0}\right)^2}{\frac{2}{a_0} \cdot 2!} = \frac{1}{a_0}$$

$$\Rightarrow \langle U\rangle = -\frac{ke^2}{a_0} \Rightarrow \langle K\rangle = \frac{ke^2}{2a_0} = \frac{\langle p^2\rangle}{2m_e} \Rightarrow$$

$$\langle p^2\rangle = \frac{m_e k e^2}{a_0} = \frac{m_e k e^2}{a_0^2} \cdot \frac{\hbar^2}{m_e k e^2} = \frac{\hbar^2}{a_0^2}$$

$\langle p\rangle = 0$ by symmetry $\Rightarrow \Delta p = \sqrt{\langle p^2\rangle} = \frac{\hbar}{a_0}$

(c) $\Delta x\, \Delta p = \frac{\sqrt{3}}{2} a_0 \cdot \frac{\hbar}{a_0} = \frac{\sqrt{3}}{2}\hbar = 0.866\hbar > \frac{1}{2}\hbar$

Problem 10

$$L^2_{op} = -\hbar^2 \left[\frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \quad ; \quad L_{z,op} = \frac{\hbar}{i} \frac{\partial}{\partial\phi}$$

$$\psi(r,\theta,\phi) = R(r) \sin^2\theta \sin\phi \cos\phi \equiv R(r)\, \Theta(\theta)\, \Phi(\phi)$$

(a) The uncertainty in L_z is zero if $L_{z,op}\psi = c\,\psi$, with c a constant

Now:

$$L_{z,op}\, \Phi = \frac{\hbar}{i} \frac{d}{d\phi} (\sin\phi \cos\phi) = \frac{\hbar}{i} [\cos^2\phi - \sin^2\phi] \neq c\, \Phi$$

therefore, uncertainty in L_z is not zero.

(b) $\langle L_z \rangle = 0$, so $\Delta L_z = \sqrt{\langle L_z^2 \rangle}$

$$L^2_{z,op} \Phi = -\hbar^2 \frac{d^2}{d\phi^2} \Phi = -\hbar^2 \frac{d}{d\phi} (\cos^2\phi - \sin^2\phi) =$$

$$= -\hbar^2 \left(-2\cos\phi\sin\phi - 2\sin\phi\cos\phi \right) = 4\hbar^2 \sin\phi\cos\phi$$

$$\Rightarrow L^2_{z,op} \Phi = 4\hbar^2 \Phi \quad \Rightarrow \quad \langle L_z^2 \rangle = 4\hbar^2$$

$$\Rightarrow \boxed{\Delta L_z = 2\hbar}$$

l is found through $L^2_{op}\psi = \hbar^2 l(l+1)\psi$. So:

$$\frac{d}{d\theta} \sin^2\theta = 2\sin\theta\cos\theta \; ; \quad \frac{d^2 \sin^2\theta}{d\theta^2} = 2(\cos^2\theta - \sin^2\theta) \quad . \text{ So:}$$

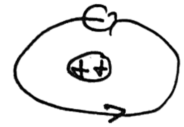
$$L^2_{op}\, \Theta(\theta)\Phi = -\hbar^2 \left[\frac{\cos\theta}{\cancel{\sin\theta}} \cdot 2\cancel{\sin\theta}\cos\theta + 2\cos^2\theta - 2\sin^2\theta - 4 \right] \Phi(\phi)$$

$$= -\hbar^2 \left[4\cos^2\theta - 2\sin^2\theta - 4\cos^2\theta - 4\sin^2\theta \right] \Phi = 6\hbar^2\, \Theta(\theta)\, \Phi(\phi)$$

Therefore, $l(l+1) = 6 \Rightarrow \boxed{l = 2}$

Problem 11

Fn the He^+ ion, $Z = 2$.

$B = \frac{\mu_0 i}{2r}$. i is the current seen by the electron 

$i = \frac{Ze}{T}$, $T = \text{period} = \frac{2\pi r}{v} \Rightarrow i = \frac{Ze}{2\pi r} v$

$$\Rightarrow B = \frac{\mu_0 Ze}{4\pi r^2} v = \frac{\mu_0}{4\pi} \frac{Ze}{r^2} \frac{n\hbar}{m_e r} = \frac{\mu_0}{4\pi} \frac{en\hbar}{m_e} \frac{Z}{r^3}$$

We have: $m_e v r = n\hbar \Rightarrow v = \frac{n\hbar}{m_e r}$

$\cancel{r_n = a_0 Z^2}$ $r_n = \frac{a_0}{Z} n^2 \Rightarrow \frac{1}{r_n^3} = \frac{Z^3}{a_0^3 n^6}$

$$\Rightarrow \boxed{B = \text{const} \,.\, Z^4}$$

So we had: $B_{int} = 0.39\,T$ for H $\Rightarrow$

$$\boxed{B_{int} = 6.24\,T \text{ for } He^+}$$

We had: $\Delta E = 2\mu_B B_{int}$,

$\Delta E = 4.53 \cdot 10^{-5}\, eV$ for H $\Rightarrow$

$$\boxed{\Delta E = 7.25 \cdot 10^{-4}\, eV \text{ for } He^+}$$