PHYSICS 2D FINAL EXAM FALL QUARTER 2023
PROF. HIRSCH

Justify all your answers to all 11 problems. Write clearly.

Time dilation; Length contraction: $\Delta t = \gamma \Delta t_0$; $L = L_0 / \gamma$; $c = 3\times 10^8 m/s$

Lorentz transformation: $x' = \gamma(x - ut)$; $y' = y$; $t' = \gamma(t - ux/c^2)$

Velocity: $v_x' = \dfrac{v_x - u}{1 - uv_x / c^2}$; $v_y' = \dfrac{v_y}{\gamma(1 - uv_x/c^2)}$; $\gamma = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

Inverse transformations: $u \to -u$, *primed* $\leftrightarrow$ *unprimed*; Doppler: $f' = f\sqrt{\dfrac{1 \pm u/c}{1 \mp u/c}}$

Momentum: $\vec{p} = \gamma m \vec{v}$; Energy: $E = \gamma mc^2$; Kinetic energy: $K = (\gamma - 1)mc^2$

$E = \sqrt{p^2c^2 + m^2c^4}$; rest energy: $E_0 = mc^2$

Electron: $m_e = 0.511 Mev/c^2$; Proton: $m_p = 938.26 Mev/c^2$; Neutron: $m_n = 939.55 Mev/c^2$

Atomic unit: $1u = 931.5 MeV/c^2$; electron volt: $1eV = 1.6\times 10^{-19} J$

Photoelectric effect: $eV_s = K_{max} = hf - \phi = hc/\lambda - \phi$; ϕ = work function

Stefan law: $I = \sigma T^4$, $\sigma = 5.67037\times 10^{-8} W/m^2\cdot K^4$; Wien's law: $\lambda_m T = 2.8978\times 10^{-3} m\cdot K$

$I(T) = \int_0^\infty I(\lambda, T)\, d\lambda$; $I = (c/4)u$; $u(\lambda, T) = N(\lambda) E_{av}(\lambda, T)$; $N(\lambda) = \dfrac{8\pi}{\lambda^4}$

Boltzmann distribution: $N(E) = Ce^{-E/kT}$; $N = \int_0^\infty N(E)\, dE$; $E_{av} = \dfrac{1}{N}\int_0^\infty EN(E)\, dE$

Classical: $E_{av} = kT$; Planck: $E_n = n\varepsilon = nhf$; $N = \sum_{n=0}^\infty N(E_n)$; $E_{av} = \dfrac{1}{N}\sum_{n=0}^\infty E_n N(E_n)$

Planck: $E_{av} = \dfrac{hc/\lambda}{e^{hc/\lambda kT} - 1}$; $hc = 1{,}240 eV\cdot nm$; $\lambda_m T = hc/4.96k$; $\sigma = \dfrac{2\pi^5 k^4}{15c^2h^3}$

Boltzmann constant: $k = (1/11{,}604) eV/K$; 1Å=1A=0.1nm

Compton scattering: $\lambda' - \lambda = \lambda_c(1 - \cos\theta)$; $\lambda_c = \dfrac{h}{m_e c} = 0.00243 nm$

double-slit interference maxima: $d\sin\theta = n\lambda$; single-slit diffraction minima: $a\sin\theta = n\lambda$

de Broglie: $\lambda = \dfrac{h}{p}$; $f = \dfrac{E}{h}$; $\omega = 2\pi f$; $k = \dfrac{2\pi}{\lambda}$; $E = \hbar\omega$; $p = \hbar k$

matter: $E = \dfrac{p^2}{2m}$ (nonrelativistic) or $E = \sqrt{p^2c^2 + m^2c^4}$ (relativistic); photons: $E = pc$

Uncertainty: $\Delta x \Delta k \sim 1$ $\Delta t \Delta\omega \sim 1$; $\Delta x \Delta p \sim \hbar$ $\Delta t \Delta E \sim \hbar$; $\Delta A = \sqrt{< A^2 > - < A >^2}$

$\hbar c = 197.3$ eV nm ; group and phase velocity : $v_g = \dfrac{d\omega}{dk}$; $v_p = \dfrac{\omega}{k}$

Schrodinger equation: $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\dfrac{\partial\Psi}{\partial t}$; $\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$

Time – independent Schrodinger equation: $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi}{\partial x^2} + U(x)\psi(x) = E\psi(x)$; $\int_{-\infty}^{\infty} dx\ \psi^*\psi = 1$

$P(x_1 < x < x_2) = \int_{x_1}^{x_2} |\psi(x)|^2 dx$; $< f(x) >= \int_{-\infty}^{\infty} |\psi(x)|^2 f(x) dx$

∞ square well: $\psi_n(x) = \sqrt{\dfrac{2}{L}}\sin(\dfrac{n\pi x}{L})$; $E_n = \dfrac{\pi^2\hbar^2 n^2}{2mL^2}$; $\dfrac{\hbar^2}{2m_e} = 0.0381 eVnm^2$ (electron)

In 2 or 3 dimensions: $-\dfrac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + U(\vec{r})\psi(\vec{r}) = E\psi(\vec{r})$

2D square well: $\Psi_{n_1 n_2}(x,y) = \Psi_{1,n_1}(x)\Psi_{2,n_2}(y)$; $E_{n_1 n_2} = \dfrac{\pi^2\hbar^2}{2m}(\dfrac{n_1^2}{L_1^2} + \dfrac{n_2^2}{L_2^2})$; $\Psi_{i,n}(w) = \sqrt{\dfrac{2}{L_i}}\sin(\dfrac{n\pi w}{L_i})$

Harmonic oscillator: $\Psi_n(x) = H_n(x)e^{-\frac{m\omega}{2\hbar}x^2}$; $E_n = (n + \dfrac{1}{2})\hbar\omega$; $E = \dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 x^2 = \dfrac{1}{2}m\omega^2 A^2$

Step potential: reflection coef : $R = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$, $T = 1 - R$; $k = \sqrt{\dfrac{2m}{\hbar^2}(E - U)}$

Tunneling: $\psi(x) \sim e^{-\alpha x}$; $T = e^{-2\alpha\Delta x}$; $T = e^{-2\int_{x_1}^{x_2}\alpha(x)dx}$; $\alpha(x) = \sqrt{\dfrac{2m[U(x) - E]}{\hbar^2}}$

Rutherford scattering: $U = \dfrac{2Ze^2}{4\pi\varepsilon_0 r}$; $e^2/(4\pi\varepsilon_0) = 1.44 eV\cdot nm = ke^2$

$b = \dfrac{Z}{K_\alpha}\dfrac{e^2}{4\pi\varepsilon_0}\cot\dfrac{1}{2}\theta$; $f_{>\theta} = nt\pi b^2$; $N(\theta) = \text{constant} \times (\dfrac{Z}{K_\alpha})^2 \times \dfrac{1}{\sin^4(\theta/2)}$

Line spectra: $\dfrac{1}{\lambda} = R(\dfrac{1}{n_0^2} - \dfrac{1}{n^2})$; $R = \dfrac{1}{91.13nm}$

Bohr atom: $r_n = (a_0/Z)n^2$; $E_n = -E_0 Z^2/n^2$; $E_0 = \dfrac{ke^2}{2a_0}$; $a_0 = \dfrac{\hbar^2}{m_e ke^2}$; $L = m_e vr = n\hbar$

$E_0 = 13.6eV$; $a_0 = 0.0529nm$

Observables, operators, eigenvalues, eigenfunctions:

$< A >= \int d^3r\psi^*(\vec{r})A_{op}\psi(\vec{r})$; if $A_{op}\psi = a\psi$ ==> $\Delta A = 0$

Spherically symmetric potential: $\Psi_{n,\ell,m_\ell}(r,\theta,\phi) = R_{n\ell}(r)Y_{\ell,m_\ell}(\theta,\phi)$; $Y_{\ell,m_\ell}(\theta,\phi) = P_\ell^{m_\ell}(\theta)e^{im_\ell\phi}$

quantum numbers: $n = 1,2,3,...$; $0 \le \ell \le n-1$; $-\ell \le m_\ell \le \ell$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$; $|L^2| = \ell(\ell+1)\hbar^2$; $L_z = m_\ell \hbar$

Radial probability density: $P(r) = r^2 |R_{n\ell}(r)|^2$; Energy: $E_n = -\frac{ke^2}{2a_0}\frac{Z^2}{n^2}$

Ground state of hydrogen-like ions: $\Psi_{1,0,0} = \frac{1}{\pi^{1/2}}(\frac{Z}{a_0})^{3/2}e^{-Zr/a_0}$; $\int_0^\infty dr\, r^n e^{-\lambda r} = \frac{n!}{\lambda^{n+1}}$

Orbital magnetic moment: $\vec{\mu} = \frac{-e}{2m_e}\vec{L}$; $\mu_z = -\mu_B m_l$; $\mu_B = \frac{e\hbar}{2m_e} = 5.79\times 10^{-5} eV/T$

Spin 1/2: $s = \frac{1}{2}$, $|\vec{S}| = \sqrt{s(s+1)}\hbar$; $S_z = m_s\hbar$; $m_s = \pm 1/2$; $\vec{\mu}_s = \frac{-e}{2m_e}g\vec{S}$; $g = 2$

Orbital+spin mag moment: $\vec{\mu} = \frac{-e}{2m_e}(\vec{L} + g\vec{S})$; Energy in mag. field: $U = -\vec{\mu}\cdot\vec{B}$

subshells: 1s, 2s, 2p, 3s, 3p, 4s, 3d, 4p, 5s, 4d, 5p, 6s, 4f, 5d, 6p, 7s, 5f, 6d

$2(2\ell+1)$ electrons per subshell. Screened electron: $E_n = (-13.6eV)Z_{eff}^2/n^2$

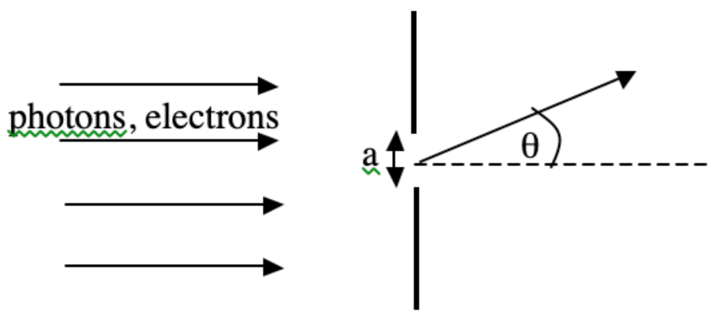
Problem 1 (6 points)
On their 20th birthday, twin B departs earth on a spaceship traveling at speed 0.6c with respect to the earth, twin A remains on earth. On their 21st birthday, as measured in each own's reference frame, they both light candles to celebrate.
(a) How old is twin B when the light from twin A's candle reaches him/her, as measured in twin B's reference frame?
(b) How old is twin A when the light from twin B's candle reaches him/her, as measured in twin A 's reference frame?
Explain all steps.

Problem 2 (6 points)
A blackbody emits maximum power per unit wavelength at wavelength 1000nm.
(a) What is its temperature, in degree Kelvin?
(b) If at the temperature found in (a), this body emits 100W/nm at wavelength 100,000nm, how much power per nm does it emit at 100,000nm if its temperature is raised by 100K?
Hint: your answer can be approximate. Use Taylor expansion if justified.

Problem 3 (6 points)

A photon incident on an electron at rest is scattered at a 60^o angle relative to the incidence direction. The wavelength of the scattered photon is the Compton wavelength, 0.00243nm.
(a) Find the kinetic energy of the scattered electron, in eV.
(b) Find the angle of the scattered electron's trajectory relative to the incidence direction. Give your answer in degrees.

Problem 4 (6 points)



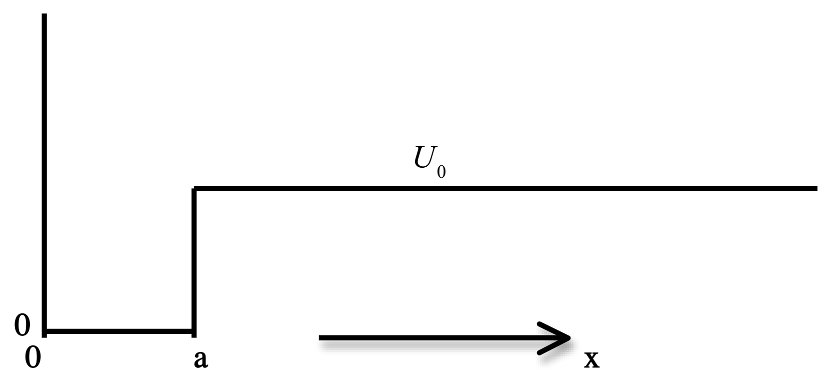
Photons with photon energy 2eV and electrons with kinetic energy 2eV are incident on a screen that is perpendicular to the beam and has a narrow slit of width $a = 0.5\mu m$ ($\mu m = 10^{-6} m$). After passing through the slit, photons and electrons emerge each with a range of angles θ relative to the incidence direction. Using the uncertainty principle (not using any other formula), estimate the range of angles θ for
(a) the photons
(b) the electrons
Give your answers in degrees.

Problem 5 (6 points)

Hydrogen atoms in a cubic container with transparent walls are moving at speed 1000m/s in random directions. Electrons are all in their ground state. Photons from a radiation source are incident through one of the walls of the container in direction perpendicular to the wall.
(a) What should be the smallest wavelength λ of the radiation so that atoms that are moving towards the wall through which the radiation is coming in will absorb photons and atoms moving away from it will not? Give an approximate value for λ and explain your answer.
(b) The atoms that absorb radiation will change their speed. Do they speed up or slow down? Calculate their speed after absorbing a photon, in m/s.

<u>Problem 6</u> **(6 points)**
There are 5 electrons in a two-dimensional square box of side length 0.56nm. Electrons don't interact with each other but have spin ½ and obey the Pauli principle. The system is in the lowest possible energy state. There is no magnetic field.
(a) Find the total energy in eV.
(b) Find the wavelength of the longest wavelength photon that this system can absorb. Give your answer in nm.
(c) Find the wavelength of the next longest wavelength photon that this system can absorb, different from the one found in (b). Give your answer in nm.
There are no selection rules for this system.

<u>Problem 7</u> **(6 points)**



An electron is in the one-dimensional potential well shown in the figure. The left wall is of infinite height. The width of the well is a=0.5nm. The potential energy is 0 for $0 < x < a$. The height of the potential well for $x > a$ is U_0 .
(a) Assume first that $U_0 = \infty$. Give the wavefunction and the energy (in eV) for the electron in the ground state.
(b) For a finite value of U_0 , the ground state energy is 1eV. Find what is the value of U_0 , in eV.
<u>Hint:</u> Use the conditions that the wavefunction and its first derivative have to be continuous at x=a. Explain all steps.

<u>Problem 8</u> **(6 points)**
The radius of the tin nucleus, with Z=50, is R=$6.666 \times 10^{-6} nm$
(a) α particles of kinetic energy 7.2 MeV are incident on a foil of tin. Find the distance of closest approach of an α particle to the center of a nucleus, d, in nm
(b) Approximate the Coulomb potential of the tin nucleus by a square barrier of height the average of the Coulomb potentials at distances R and d from the center of the nucleus. The potential energy is negative inside the nucleus, i.e. for r<R.
Calculate the probability that an α particle incident on a tin nucleus will penetrate the Coulomb barrier and enter the nucleus. If 100 million α particle are incident on nuclei reaching their distance of closest approach, how many will end up inside nuclei?

Problem 9 (6 points)

An electron in a one-dimensional atom has potential energy $U(x) = -ke^2/x$ (x>0). We found by solving the one-dimensional Schrodinger equation that its ground state wave function is

$$\psi(x) = Cxe^{-x/a_0} \quad x>0, 0 \text{ for } x<0$$

with C a constant and a_0 the Bohr radius. Its ground state energy is $E = -ke^2/(2a_0)$, same as in three dimensions.

(a) Calculate the uncertainty in the position of the electron, Δx, expressed as a number times a_0.

(b) Calculate the uncertainty in its momentum, Δp, expressed as a number times $(\hbar / a_0)$. You may use the fact that just like in 3 dimensions the average kinetic energy is minus half the average potential energy. You may not assume that the average potential energy is the same as in 3 dimensions. Justify all steps.

(c) Verify whether the uncertainty theorem is satisfied, namely whether $\Delta x \Delta p \geq \hbar / 2$

Problem 10 (6 points)

The angular momentum operators are given by the formulae

$$L_{op}^2 = -\hbar^2 [\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}] \quad ; \quad L_{z,op} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

An electron in a hydrogen atom is in a stationary state described by the wavefunction

$$\psi(r,\theta,\phi) = R(r)\sin^2\theta\sin\phi\cos\phi$$

(a) Explain how you can tell that the uncertainty in L_z for this electron is not zero.

(b) Calculate the uncertainty in L_z. You may use the fact that $< L_z >= 0$ for this wavefunction without proving it.

(c) By applying an operator to this wavefunction find the quantum number ℓ. Justify all steps. Use that $1 = \sin^2\theta + \cos^2\theta$

Problem 11 (6 points)

For a hydrogen atom with an electron in the 2p state, we estimated that the internal magnetic field seen by the electron due to its orbital motion around the proton was $B_{\text{int}} = 0.39T$, and the resulting fine structure splitting was $\Delta E = 4.53 \times 10^{-5} eV$.

Using the same reasoning, find B_{int} and ΔE for an electron in the 2p state of the He^+ ion.
Hints: Express your quantities in terms of Z (Z=2 here) to extract the Z-dependence of the quantities you calculate, then you can obtain the answers from the hydrogen answers. The magnetic field at the center of a circular loop of radius r through which a current i flows is $B = \mu_o i / (2r)$. Remember that i=current has units charge divided by time. Use Bohr atom relations. Justify all steps.