8 Boltzmann Equation : Worked Examples

(8.1) Consider a monatomic ideal gas in the presence of a temperature gradient $\boldsymbol{\nabla} T$. Answer the following questions within the framework of the relaxation time approximation to the Boltzmann equation.

(a) Compute the particle current $\boldsymbol{j}$ and show that it vanishes.

(b) Compute the 'energy squared' current,

$$\boldsymbol{j}_{\varepsilon^2} = \int d^3\!p\, \varepsilon^2 \boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{p},t) \quad .$$

(c) Suppose the gas is diatomic, so $c_p = \frac{7}{2}k_{\rm B}$. Show explicitly that the particle current $\boldsymbol{j}$ is zero. *Hint: To do this, you will have to understand the derivation of ch. 8 of the lecture notes and how this changes when the gas is diatomic. You may assume* $\mathcal{Q}_{\alpha\beta} = \boldsymbol{F} = 0$.

Solution :

(a) Under steady state conditions, the solution to the Boltzmann equation is $f = f^0 + \delta f$, where f^0 is the equilibrium distribution and

$$\delta f = -\frac{\tau f^0}{k_{\rm B}T} \cdot \frac{\varepsilon - c_p T}{T}\, \boldsymbol{v} \cdot \boldsymbol{\nabla} T \quad .$$

For the monatomic ideal gas, $c_p = \frac{5}{2}k_{\rm B}$. The particle current is

$$\begin{aligned} j^\alpha &= \int d^3\!p\, v^\alpha\, \delta f \\ &= -\frac{\tau}{k_{\rm B}T^2}\int d^3\!p\, f^0(\boldsymbol{p})\, v^\alpha\, v^\beta \left(\varepsilon - \tfrac{5}{2}k_{\rm B}T\right) \frac{\partial T}{\partial x^\beta} \\ &= -\frac{2n\tau}{3mk_{\rm B}T^2}\, \frac{\partial T}{\partial x^\alpha} \left\langle \varepsilon\big(\varepsilon - \tfrac{5}{2}k_{\rm B}T\big)\right\rangle \quad , \end{aligned}$$

where the average over momentum/velocity is converted into an average over the energy distribution,

$$\tilde{P}(\varepsilon) = 4\pi v^2\, \frac{dv}{d\varepsilon}\, P_{\rm M}(v) = \tfrac{2}{\sqrt{\pi}}(k_{\rm B}T)^{-3/2}\, \varepsilon^{1/2}\, \phi(\varepsilon)\, e^{-\varepsilon/k_{\rm B}T} \quad .$$

As discussed in the lecture notes, the average of a homogeneous function of ε under this distribution is given by

$$\langle \varepsilon^\alpha \rangle = \tfrac{2}{\sqrt{\pi}}\,\Gamma\big(\alpha + \tfrac{3}{2}\big)\,(k_{\rm B}T)^\alpha \quad .$$

Thus,

$$\left\langle \varepsilon\big(\varepsilon - \tfrac{5}{2}k_{\rm B}T\big)\right\rangle = \tfrac{2}{\sqrt{\pi}}\,(k_{\rm B}T)^2 \left\{ \Gamma\big(\tfrac{7}{2}\big) - \tfrac{5}{2}\,\Gamma\big(\tfrac{5}{2}\big)\right\} = 0 \quad .$$

(b) Now we must compute

$$\begin{aligned} j^\alpha_{\varepsilon^2} &= \int d^3\!p\, v^\alpha\, \varepsilon^2\, \delta f \\ &= -\frac{2n\tau}{3mk_{\rm B}T^2}\, \frac{\partial T}{\partial x^\alpha}\, \left\langle \varepsilon^3\big(\varepsilon - \tfrac{5}{2}k_{\rm B}T\big)\right\rangle \quad . \end{aligned}$$

We then have

$$\big\langle \varepsilon^3 \big(\varepsilon - \tfrac{5}{2}k_{\rm B}T\big)\big\rangle = \tfrac{2}{\sqrt{\pi}}\,(k_{\rm B}T)^4 \Big\{\Gamma\big(\tfrac{11}{2}\big) - \tfrac{5}{2}\,\Gamma\big(\tfrac{9}{2}\big)\Big\} = \tfrac{105}{2}\,(k_{\rm B}T)^4$$

and so

$$\boldsymbol{j}_{\varepsilon^2} = -\frac{35\,n\tau k_{\rm B}}{m}\,(k_{\rm B}T)^2\,\boldsymbol{\nabla}T \quad .$$

(c) For diatomic gases in the presence of a temperature gradient, the solution to the linearized Boltzmann equation in the relaxation time approximation is

$$\delta f = -\frac{\tau\, f^0}{k_{\rm B}T}\cdot\frac{\varepsilon(\Gamma) - c_p T}{T}\,\boldsymbol{v}\cdot\boldsymbol{\nabla}T \quad ,$$

where

$$\varepsilon(\Gamma) = \varepsilon_{\rm tr} + \varepsilon_{\rm rot} = \tfrac{1}{2}m\boldsymbol{v}^2 + \frac{\mathsf{L}_1^2 + \mathsf{L}_2^2}{2I} \quad ,$$

where $\mathsf{L}_{1,2}$ are components of the angular momentum about the instantaneous body-fixed axes, with $I \equiv I_1 = I_2 \gg I_3$. We assume the rotations about axes 1 and 2 are effectively classical, so equipartition gives $\langle\varepsilon_{\rm rot}\rangle = 2\times\frac{1}{2}k_{\rm B} = k_{\rm B}$. We still have $\langle\varepsilon_{\rm tr}\rangle = \frac{3}{2}k_{\rm B}$. Now in the derivation of the factor $\varepsilon(\varepsilon - c_p T)$ above, the first factor of ε came from the $v^\alpha v^\beta$ term, so this is translational kinetic energy. Therefore, with $c_p = \frac{7}{2}k_{\rm B}$ now, we must compute

$$\big\langle \varepsilon_{\rm tr}\big(\varepsilon_{\rm tr} + \varepsilon_{\rm rot} - \tfrac{7}{2}k_{\rm B}T\big)\big\rangle = \big\langle \varepsilon_{\rm tr}\big(\varepsilon_{\rm tr} - \tfrac{5}{2}k_{\rm B}T\big)\big\rangle = 0 \quad .$$

So again the particle current vanishes.

Note added :

It is interesting to note that there is no particle current flowing in response to a temperature gradient when τ is energy-independent. This is a consequence of the fact that the pressure gradient $\boldsymbol{\nabla}p$ vanishes. Newton's Second Law for the fluid says that $nm\dot{\boldsymbol{V}} + \boldsymbol{\nabla}p = 0$, to lowest relevant order. With $\boldsymbol{\nabla}p \neq 0$, the fluid will accelerate. In a pipe, for example, eventually a steady state is reached where the flow is determined by the fluid viscosity, which is one of the terms we just dropped. (This is called *Poiseuille flow*.) When p is constant, the local equilibrium distribution is

$$f^0(\boldsymbol{r},\boldsymbol{p}) = \frac{p/k_{\rm B}T}{(2\pi m k_{\rm B}T)^{3/2}}\,e^{-\boldsymbol{p}^2/2mk_{\rm B}T} \quad ,$$

where $T = T(\boldsymbol{r})$. We then have

$$f(\boldsymbol{r},\boldsymbol{p}) = f^0(\boldsymbol{r} - \boldsymbol{v}\tau, \boldsymbol{p}) \quad ,$$

which says that no new collisions happen for a time τ after a given particle thermalizes. *I.e.* we evolve the streaming terms for a time τ. Expanding, we have

$$\begin{aligned} f &= f^0 - \frac{\tau\boldsymbol{p}}{m}\cdot\frac{\partial f^0}{\partial\boldsymbol{r}} + \ldots \\ &= \bigg\{1 - \frac{\tau}{2k_{\rm B}T^2}\,\big(\varepsilon(\boldsymbol{p}) - \tfrac{5}{2}k_{\rm B}T\big)\,\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{\nabla}T + \ldots\bigg\}\, f^0(\boldsymbol{r},\boldsymbol{p}) \quad , \end{aligned}$$

which leads to $\boldsymbol{j} = 0$, assuming the relaxation time τ is energy-independent.

When the flow takes place in a restricted geometry, a dimensionless figure of merit known as the *Knudsen number*, $\mathsf{Kn} = \ell/L$, where ℓ is the mean free path and L is the characteristic linear dimension associated with the geometry. For $\mathsf{Kn} \ll 1$, our Boltzmann transport calculations of quantities like κ, η, and ζ hold, and we may apply the Navier-Stokes equations[1]. In the opposite limit $\mathsf{Kn} \gg 1$, the boundary conditions on the distribution are crucial. Consider, for example, the case $\ell = \infty$. Suppose we have ideal gas flow in a cylinder whose symmetry axis is $\hat{\boldsymbol{x}}$.

[1] These equations may need to be supplemented by certain conditions which apply in the vicinity of solid boundaries.

Particles with $v_x > 0$ enter from the left, and particles with $v_x < 0$ enter from the right. Their respective velocity distributions are

$$P_j(\boldsymbol{v}) = n_j \left(\frac{m}{2\pi k_{\rm B} T_j}\right)^{3/2} e^{-m\boldsymbol{v}^2/2k_{\rm B}T_j} \quad ,$$

where $j = \text{L}$ or R. The average current is then

$$\begin{aligned} j_x &= \int d^3v \, \Big\{ n_{\rm L} \, v_x \, P_{\rm L}(\boldsymbol{v}) \, \Theta(v_x) + n_{\rm R} \, v_x \, P_{\rm R}(\boldsymbol{v}) \, \Theta(-v_x) \Big\} \\ &= n_{\rm L} \sqrt{\frac{2k_{\rm B}T_{\rm L}}{m}} - n_{\rm R} \sqrt{\frac{2k_{\rm B}T_{\rm R}}{m}} \quad . \end{aligned}$$

(8.2) Consider a classical gas of charged particles in the presence of a magnetic field $\boldsymbol{B}$. The Boltzmann equation is then given by

$$\frac{\varepsilon - h}{k_{\rm B}T^2}\, f^0\, \boldsymbol{v} \cdot \boldsymbol{\nabla} T - \frac{e}{mc}\, \boldsymbol{v} \times \boldsymbol{B} \cdot \frac{\partial\, \delta f}{d\boldsymbol{v}} = \left(\frac{df}{dt}\right)_{\rm coll} \quad .$$

Consider the case where $T = T(x)$ and $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Making the relaxation time approximation, show that a solution to the above equation exists in the form $\delta f = \boldsymbol{v} \cdot \boldsymbol{A}(\varepsilon)$, where $\boldsymbol{A}(\varepsilon)$ is a vector-valued function of $\varepsilon(\boldsymbol{v}) = \frac{1}{2}m\boldsymbol{v}^2$ which lies in the (x, y) plane. Find the energy current $\boldsymbol{j}_\varepsilon$. Interpret your result physically.

`Solution` : We'll use index notation and the Einstein summation convention for ease of presentation. Recall that the curl is given by $(\boldsymbol{A} \times \boldsymbol{B})_\mu = \epsilon_{\mu\nu\lambda}\, A_\nu\, B_\lambda$. We write $\delta f = v_\mu\, A_\mu(\varepsilon)$, and compute

$$\begin{aligned} \frac{\partial\, \delta f}{\partial v_\lambda} &= A_\lambda + v_\alpha\, \frac{\partial A_\alpha}{\partial v_\lambda} \\ &= A_\lambda + m v_\lambda\, v_\alpha\, \frac{\partial A_\alpha}{\partial \varepsilon} \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \boldsymbol{v} \times \boldsymbol{B} \cdot \frac{\partial\, \delta f}{\partial \boldsymbol{v}} &= \epsilon_{\mu\nu\lambda}\, v_\mu\, B_\nu\, \frac{\partial\, \delta f}{\partial v_\lambda} \\ &= \epsilon_{\mu\nu\lambda}\, v_\mu\, B_\nu \left(A_\lambda + m v_\lambda\, v_\alpha\, \frac{\partial A_\alpha}{\partial \varepsilon} \right) \\ &= \epsilon_{\mu\nu\lambda}\, v_\mu\, B_\nu\, A_\lambda \quad . \end{aligned}$$

We then have

$$\frac{\varepsilon - h}{k_{\rm B}T^2}\, f^0\, v_\mu\, \partial_\mu T = \frac{e}{mc}\, \epsilon_{\mu\nu\lambda}\, v_\mu\, B_\nu\, A_\lambda - \frac{v_\mu\, A_\mu}{\tau} \quad .$$

Since this must be true for all $\boldsymbol{v}$, we have

$$A_\mu - \frac{eB\tau}{mc}\, \epsilon_{\mu\nu\lambda}\, n_\nu\, A_\lambda = -\frac{(\varepsilon - h)\, \tau}{k_{\rm B}T^2}\, f^0\, \partial_\mu T \quad ,$$

where $\boldsymbol{B} \equiv B\, \hat{\boldsymbol{n}}$. It is conventional to define the *cyclotron frequency*, $\omega_{\rm c} = eB/mc$, in which case

$$\left(\delta_{\mu\nu} + \omega_{\rm c}\tau\, \epsilon_{\mu\nu\lambda}\, n_\lambda \right) A_\nu = X_\mu \quad ,$$

where $\boldsymbol{X} = -(\varepsilon - h)\, \tau f^0\, \boldsymbol{\nabla} T / k_{\rm B}T^2$. So we must invert the matrix

$$M_{\mu\nu} = \delta_{\mu\nu} + \omega_{\rm c}\tau\, \epsilon_{\mu\nu\lambda}\, n_\lambda \quad .$$

To do so, we make the *Ansatz*,

$$M^{-1}_{\nu\sigma} = A\, \delta_{\nu\sigma} + B\, n_\nu\, n_\sigma + C\, \epsilon_{\nu\sigma\rho}\, n_\rho \quad ,$$

and we determine the constants A, B, and C by demanding

$$\begin{aligned} M_{\mu\nu}\, M^{-1}_{\nu\sigma} &= \left(\delta_{\mu\nu} + \omega_{\rm c}\tau\, \epsilon_{\mu\nu\lambda}\, n_\lambda \right)\left(A\, \delta_{\nu\sigma} + B\, n_\nu\, n_\sigma + C\, \epsilon_{\nu\sigma\rho}\, n_\rho \right) \\ &= \left(A - C\, \omega_{\rm c}\, \tau \right) \delta_{\mu\sigma} + \left(B + C\, \omega_{\rm c}\, \tau \right) n_\mu\, n_\sigma + \left(C + A\, \omega_{\rm c}\, \tau \right) \epsilon_{\mu\sigma\rho}\, n_\rho \equiv \delta_{\mu\sigma} \quad . \end{aligned}$$

Here we have used the result

$$\epsilon_{\mu\nu\lambda}\, \epsilon_{\nu\sigma\rho} = \epsilon_{\nu\lambda\mu}\, \epsilon_{\nu\sigma\rho} = \delta_{\lambda\sigma}\, \delta_{\mu\rho} - \delta_{\lambda\rho}\, \delta_{\mu\sigma} \quad ,$$

as well as the fact that $\hat{\boldsymbol{n}}$ is a unit vector: $n_\mu\, n_\mu = 1$. We can now read off the results:

$$A - C\, \omega_{\rm c}\tau = 1 \quad , \quad B + C\, \omega_{\rm c}\tau = 0 \quad , \quad C + A\, \omega_{\rm c}\tau = 0 \quad ,$$

which entail

$$A = \frac{1}{1+\omega_{\rm c}^2\tau^2} \quad , \quad B = \frac{\omega_{\rm c}^2\tau^2}{1+\omega_{\rm c}^2\tau^2} \quad , \quad C = -\frac{\omega_{\rm c}\tau}{1+\omega_{\rm c}^2\tau^2} \quad .$$

So we can now write

$$A_\mu = M^{-1}_{\mu\nu}\, X_\nu = \frac{\delta_{\mu\nu} + \omega_{\rm c}^2\tau^2\, n_\mu\, n_\nu - \omega_{\rm c}\tau\, \epsilon_{\mu\nu\lambda}\, n_\lambda}{1+\omega_{\rm c}^2\tau^2}\; X_\nu \quad .$$

The α-component of the energy current is

$$j^\alpha_\varepsilon = \int\!\frac{d^3\!p}{h^3}\, v_\alpha\, v_\mu\, \varepsilon\, A_\mu(\varepsilon) = \frac{2}{3m}\!\int\!\frac{d^3\!p}{h^3}\, \varepsilon^2\, A_\alpha(\varepsilon) \quad ,$$

where we have replaced $v_\alpha\, v_\mu \to \frac{2}{3m}\,\varepsilon\,\delta_{\alpha\mu}$. Next, we use

$$\frac{2}{3m}\!\int\!\frac{d^3\!p}{h^3}\, \varepsilon^2\, X_\nu = -\frac{5\tau}{3m}\, k_{\rm B}^2 T\, \frac{\partial T}{\partial x_\nu} \quad ,$$

hence

$$\boldsymbol{j}_\varepsilon = -\frac{5\tau}{3m}\, \frac{k_{\rm B}^2 T}{1+\omega_{\rm c}^2\tau^2} \Big(\boldsymbol{\nabla} T + \omega_{\rm c}^2\tau^2\, \hat{\boldsymbol{n}}\, (\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla} T) + \omega_{\rm c}\tau\, \hat{\boldsymbol{n}}\times\boldsymbol{\nabla} T \Big) \quad .$$

We are given that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and $\boldsymbol{\nabla} T = T'(x)\,\hat{\boldsymbol{x}}$. We see that the energy current $\boldsymbol{j}_\varepsilon$ is flowing both along $-\hat{\boldsymbol{x}}$ and along $-\hat{\boldsymbol{y}}$. Why does heat flow along $\hat{\boldsymbol{y}}$? It is because the particles are charged, and as they individually flow along $-\hat{\boldsymbol{x}}$, there is a Lorentz force in the $-\hat{\boldsymbol{y}}$ direction, so the energy flows along $-\hat{\boldsymbol{y}}$ as well.

(8.3) Consider one dimensional motion according to the equation

$$\dot{p} + \gamma p = \eta(t) \quad ,$$

and compute the average $\langle p^4(t) \rangle$. You should assume that

$$\langle \eta(s_1)\, \eta(s_2)\, \eta(s_3)\, \eta(s_4) \rangle = \phi(s_1 - s_2)\, \phi(s_3 - s_4) + \phi(s_1 - s_3)\, \phi(s_2 - s_4) + \phi(s_1 - s_4)\, \phi(s_2 - s_3)$$

where $\phi(s) = \Gamma\, \delta(s)$. You may further assume that $p(0) = 0$.

Solution :

Integrating the Langevin equation, we have

$$p(t) = \int_0^t dt_1\, e^{-\gamma(t-t_1)}\, \eta(t_1) \quad .$$

Raising this to the fourth power and taking the average, we have

$$\begin{aligned}\langle p^4(t) \rangle &= \int_0^t dt_1\, e^{-\gamma(t-t_1)} \int_0^t dt_2\, e^{-\gamma(t-t_2)} \int_0^t dt_3\, e^{-\gamma(t-t_3)} \int_0^t dt_4\, e^{-\gamma(t-t_4)}\, \langle \eta(t_1)\, \eta(t_2)\, \eta(t_3)\, \eta(t_4) \rangle \\ &= 3\Gamma^2 \int_0^t dt_1\, e^{-2\gamma(t-t_1)} \int_0^t dt_2\, e^{-2\gamma(t-t_2)} = \frac{3\,\Gamma^2}{4\,\gamma^2} \left(1 - e^{-2\gamma t}\right)^2 \quad .\end{aligned}$$

We have here used the fact that the three contributions to the average of the product of the four η's each contribute the same amount to $\langle p^4(t) \rangle$. Recall $\Gamma = 2M\gamma k_B T$, where M is the mass of the particle. Note that

$$\langle p^4(t) \rangle = 3 \langle p^2(t) \rangle^2 \quad .$$

(8.4) A photon gas in equilibrium is described by the distribution function

$$f^0(\boldsymbol{p}) = \frac{2}{e^{cp/k_\text{B}T} - 1} \quad ,$$

where the factor of 2 comes from summing over the two independent polarization states.

(a) Consider a photon gas (in three dimensions) slightly out of equilibrium, but in steady state under the influence of a temperature gradient $\boldsymbol{\nabla}T$. Write $f = f^0 + \delta f$ and write the Boltzmann equation in the relaxation time approximation. Remember that $\varepsilon(\boldsymbol{p}) = cp$ and $\boldsymbol{v} = \frac{\partial\varepsilon}{\partial\boldsymbol{p}} = c\hat{\boldsymbol{p}}$, so the speed is always c.

(b) What is the formal expression for the energy current, expressed as an integral of something times the distribution f?

(c) Compute the thermal conductivity κ. It is OK for your expression to involve *dimensionless* integrals.

Solution :

(a) We have

$$df^0 = -\frac{2cp\, e^{\beta cp}}{(e^{\beta cp} - 1)^2}\, d\beta = \frac{2cp\, e^{\beta cp}}{(e^{\beta cp} - 1)^2}\, \frac{dT}{k_\text{B}T^2} \quad .$$

The steady state Boltzmann equation is $\boldsymbol{v} \cdot \frac{\partial f^0}{\partial \boldsymbol{r}} = \left(\frac{\partial f}{\partial t}\right)_\text{coll}$, hence with $\boldsymbol{v} = c\hat{\boldsymbol{p}}$,

$$\frac{2\, c^2\, e^{cp/k_\text{B}T}}{(e^{cp/k_\text{B}T} - 1)^2}\, \frac{1}{k_\text{B}T^2}\, \boldsymbol{p} \cdot \boldsymbol{\nabla}T = -\frac{\delta f}{\tau} \quad .$$

(b) The energy current is given by

$$\boldsymbol{j}_\varepsilon(\boldsymbol{r}) = \int \frac{d^3\!p}{h^3}\, c^2 \boldsymbol{p}\, f(\boldsymbol{p}, \boldsymbol{r}) \quad .$$

(c) Integrating, we find

$$\begin{aligned}
\kappa &= \frac{2c^4\tau}{3h^3 k_\text{B}T^2} \int d^3\!p\, \frac{p^2\, e^{cp/k_\text{B}T}}{(e^{cp/k_\text{B}T} - 1)^2} \\
&= \frac{8\pi k_\text{B}\tau}{3c} \left(\frac{k_\text{B}T}{hc}\right)^3 \int\limits_0^\infty ds\, \frac{s^4\, e^s}{(e^s - 1)^2} \\
&= \frac{4k_\text{B}\tau}{3\pi^2 c} \left(\frac{k_\text{B}T}{hc}\right)^3 \int\limits_0^\infty ds\, \frac{s^3}{e^s - 1} \quad ,
\end{aligned}$$

where we simplified the integrand somewhat using integration by parts. The integral may be computed in closed form:

$$\mathcal{I}_n = \int\limits_0^\infty ds\, \frac{s^n}{e^s - 1} = \Gamma(n+1)\,\zeta(n+1) \quad \Rightarrow \quad \mathcal{I}_3 = \frac{\pi^4}{15} \quad ,$$

and therefore

$$\kappa = \frac{\pi^2 k_\text{B}\tau}{45\, c} \left(\frac{k_\text{B}T}{hc}\right)^3 \quad .$$

(8.5) Suppose the relaxation time is energy-dependent, with $\tau(\varepsilon) = \tau_0\, e^{-\varepsilon/\varepsilon_0}$. Compute the particle current $\boldsymbol{j}$ and energy current $\boldsymbol{j}_\varepsilon$ flowing in response to a temperature gradient $\boldsymbol{\nabla} T$.

Solution :

Now we must compute

$$\begin{aligned}
\begin{Bmatrix} j^\alpha \\ j^\alpha_\varepsilon \end{Bmatrix} &= \int\! d^3\!p \begin{Bmatrix} v^\alpha \\ \varepsilon\, v^\alpha \end{Bmatrix} \delta f \\
&= -\frac{2n}{3mk_{\rm B}T^2}\,\frac{\partial T}{\partial x^\alpha}\,\Big\langle \tau(\varepsilon) \begin{Bmatrix} \varepsilon \\ \varepsilon^2 \end{Bmatrix} \big(\varepsilon - \tfrac{5}{2}k_{\rm B}T\big)\Big\rangle \quad ,
\end{aligned}$$

where $\tau(\varepsilon) = \tau_0\, e^{-\varepsilon/\varepsilon_0}$. We find

$$\begin{aligned}
\big\langle e^{-\varepsilon/\varepsilon_0}\, \varepsilon^\alpha \big\rangle &= \tfrac{2}{\sqrt{\pi}}\,(k_{\rm B}T)^{-3/2} \int\limits_0^\infty \! d\varepsilon\; \varepsilon^{\alpha+\frac{1}{2}} e^{-\varepsilon/k_{\rm B}T}\, e^{-\varepsilon/\varepsilon_0} \\
&= \tfrac{2}{\sqrt{\pi}}\,\Gamma\big(\alpha+\tfrac{3}{2}\big)(k_{\rm B}T)^\alpha \bigg(\frac{\varepsilon_0}{\varepsilon_0 + k_{\rm B}T}\bigg)^{\alpha+\frac{3}{2}} \quad .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big\langle e^{-\varepsilon/\varepsilon_0}\, \varepsilon \big\rangle &= \tfrac{3}{2}\, k_{\rm B}T \,\bigg(\frac{\varepsilon_0}{\varepsilon_0 + k_{\rm B}T}\bigg)^{5/2} \\
\big\langle e^{-\varepsilon/\varepsilon_0}\, \varepsilon^2 \big\rangle &= \tfrac{15}{4}\, (k_{\rm B}T)^2 \,\bigg(\frac{\varepsilon_0}{\varepsilon_0 + k_{\rm B}T}\bigg)^{7/2} \\
\big\langle e^{-\varepsilon/\varepsilon_0}\, \varepsilon^3 \big\rangle &= \tfrac{105}{8}\, (k_{\rm B}T)^3 \,\bigg(\frac{\varepsilon_0}{\varepsilon_0 + k_{\rm B}T}\bigg)^{9/2}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{j} &= \frac{5n\tau_0 k_{\rm B}^2 T}{2m}\,\frac{\varepsilon_0^{5/2}}{(\varepsilon_0 + k_{\rm B}T)^{7/2}}\,\boldsymbol{\nabla} T \\
\boldsymbol{j}_\varepsilon &= -\frac{5n\tau_0 k_{\rm B}^2 T}{4m}\,\bigg(\frac{\varepsilon_0}{\varepsilon_0 + k_{\rm B}T}\bigg)^{7/2} \bigg(\frac{2\varepsilon_0 - 5k_{\rm B}T}{\varepsilon_0 + k_{\rm B}T}\bigg)\boldsymbol{\nabla} T \quad .
\end{aligned}$$

The previous results are obtained by setting $\varepsilon_0 = \infty$ and $\tau_0 = 1/\sqrt{2}\, n\bar{v}\sigma$. Note the strange result that κ becomes negative for $k_{\rm B}T > \frac{2}{5}\varepsilon_0$.

(8.6) Use the linearized Boltzmann equation to compute the bulk viscosity ζ of an ideal gas.

(a) Consider first the case of a monatomic ideal gas. Show that $\zeta = 0$ within this approximation. Will your result change if the scattering time is energy-dependent?

(b) Compute ζ for a diatomic ideal gas.

Solution :

According to the lecture notes, the solution to the linearized Boltzmann equation in the relaxation time approximation is

$$\delta f = -\frac{\tau f^0}{k_{\rm B}T}\left\{ m v^\alpha v^\beta \, \frac{\partial V_\alpha}{\partial x^\beta} - \big(\varepsilon_{\rm tr} + \varepsilon_{\rm rot}\big)\, \frac{k_{\rm B}}{c_V}\, \nabla\!\cdot\!\boldsymbol{V}\right\} \quad .$$

We also have

$$\mathsf{Tr}\, \Pi = nm\, \langle \boldsymbol{v}^2\rangle = 2n\, \langle \varepsilon_{\rm tr}\rangle = 3p - 3\zeta\, \nabla\!\cdot\!\boldsymbol{V} \quad .$$

We then compute $\mathsf{Tr}\,\Pi$:

$$\begin{aligned}\mathsf{Tr}\,\Pi &= 2n\,\langle \varepsilon_{\rm tr}\rangle = 3p - 3\zeta\, \nabla\!\cdot\!\boldsymbol{V}\\ &= 2n\!\int\! d\varGamma\, (f^0 + \delta f)\, \varepsilon_{\rm tr}\end{aligned}$$

The f^0 term yields a contribution $3nk_{\rm B}T = 3p$ in all cases, which agrees with the first term on the RHS of the equation for $\mathsf{Tr}\,\Pi$. Therefore

$$\zeta\, \nabla\!\cdot\!\boldsymbol{V} = -\tfrac{2}{3} n\!\int\! d\varGamma\, \delta f\, \varepsilon_{\rm tr} \quad .$$

(a) For the monatomic gas, $\varGamma = \{p_x, p_y, p_z\}$. We then have

$$\begin{aligned}\zeta \nabla\!\cdot\!\boldsymbol{V} &= \frac{2n\tau}{3k_{\rm B}T}\!\int\! d^3\!p\, f^0(\boldsymbol{p})\, \varepsilon \left\{ m v^\alpha v^\beta\, \frac{\partial V_\alpha}{\partial x^\beta} - \frac{\varepsilon}{c_V/k_{\rm B}}\, \nabla\!\cdot\!\boldsymbol{V}\right\}\\ &= \frac{2n\tau}{3k_{\rm B}T}\left\langle \Big(\tfrac{2}{3} - \tfrac{k_{\rm B}}{c_V}\Big)\, \varepsilon \right\rangle \nabla\!\cdot\!\boldsymbol{V} = 0 \quad .\end{aligned}$$

Here we have replaced $m v^\alpha v^\beta \to \frac{1}{3} m\boldsymbol{v}^2 = \frac{2}{3}\varepsilon_{\rm tr}$ under the integral. If the scattering time is energy dependent, then we put $\tau(\varepsilon)$ inside the energy integral when computing the average, but this does not affect the final result: $\zeta = 0$.

(b) Now we must include the rotational kinetic energy in the expression for δf, and we have $c_V = \frac{5}{2}k_{\rm B}$. Thus,

$$\begin{aligned}\zeta \nabla\!\cdot\!\boldsymbol{V} &= \frac{2n\tau}{3k_{\rm B}T}\!\int\! d\varGamma\, f^0(\varGamma)\, \varepsilon_{\rm tr} \left\{ m v^\alpha v^\beta\, \frac{\partial V_\alpha}{\partial x^\beta} - \big(\varepsilon_{\rm tr} + \varepsilon_{\rm rot}\big)\, \frac{k_{\rm B}}{c_V}\, \nabla\!\cdot\!\boldsymbol{V}\right\}\\ &= \frac{2n\tau}{3k_{\rm B}T}\left\langle \tfrac{2}{3}\varepsilon_{\rm tr}^2 - \tfrac{k_{\rm B}}{c_V}(\varepsilon_{\rm tr} + \varepsilon_{\rm rot})\varepsilon_{\rm tr}\right\rangle \nabla\!\cdot\!\boldsymbol{V} \quad ,\end{aligned}$$

and therefore

$$\zeta = \frac{2n\tau}{3k_{\rm B}T}\,\big\langle \tfrac{4}{15}\,\varepsilon_{\rm tr}^2 - \tfrac{2}{5}k_{\rm B}T\,\varepsilon_{\rm tr}\big\rangle = \tfrac{4}{15} n\tau k_{\rm B}T \quad .$$

(8.7) Consider a two-dimensional gas of particles with dispersion $\varepsilon(\boldsymbol{k}) = J\boldsymbol{k}^2$, where $\boldsymbol{k}$ is the wavevector. The particles obey photon statistics, so $\mu = 0$ and the equilibrium distribution is given by

$$f^0(\boldsymbol{k}) = \frac{1}{e^{\varepsilon(\boldsymbol{k})/k_B T} - 1} \quad .$$

(a) Writing $f = f^0 + \delta f$, solve for $\delta f(\boldsymbol{k})$ using the steady state Boltzmann equation in the relaxation time approximation,

$$\boldsymbol{v} \cdot \frac{\partial f^0}{\partial \boldsymbol{r}} = -\frac{\delta f}{\tau} \quad .$$

Work to lowest order in $\boldsymbol{\nabla} T$. Remember that $\boldsymbol{v} = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \boldsymbol{k}}$ is the velocity.

(b) Show that $\boldsymbol{j} = -\lambda\, \boldsymbol{\nabla} T$, and find an expression for λ. Represent any integrals you cannot evaluate as dimensionless expressions.

(c) Show that $\boldsymbol{j}_\varepsilon = -\kappa\, \boldsymbol{\nabla} T$, and find an expression for κ. Represent any integrals you cannot evaluate as dimensionless expressions.

Solution :

(a) We have

$$\begin{aligned}\delta f = -\tau\, \boldsymbol{v} \cdot \frac{\partial f^0}{\partial \boldsymbol{r}} &= -\tau\, \boldsymbol{v} \cdot \boldsymbol{\nabla} T\, \frac{\partial f^0}{\partial T} \\ &= -\frac{2\tau}{\hbar}\, \frac{J^2 k^2}{k_B T^2}\, \frac{e^{\varepsilon(\boldsymbol{k})/k_B T}}{\big(e^{\varepsilon(\boldsymbol{k})/k_B T} - 1\big)^2}\, \boldsymbol{k} \cdot \boldsymbol{\nabla} T\end{aligned}$$

(b) The particle current is

$$\begin{aligned}j^\mu &= \frac{2J}{\hbar} \int \frac{d^2k}{(2\pi)^2}\, k^\mu\, \delta f(\boldsymbol{k}) = -\lambda\, \frac{\partial T}{\partial x^\mu} \\ &= -\frac{4\tau}{\hbar^2}\, \frac{J^3}{k_B T^2}\, \frac{\partial T}{\partial x^\nu} \int \frac{d^2k}{(2\pi)^2}\, k^2\, k^\mu\, k^\nu\, \frac{e^{Jk^2/k_B T}}{\big(e^{Jk^2/k_B T} - 1\big)^2}\end{aligned}$$

We may now send $k^\mu k^\nu \to \frac{1}{2} k^2 \delta^{\mu\nu}$ under the integral. We then read off

$$\begin{aligned}\lambda &= \frac{2\tau}{\hbar^2}\, \frac{J^3}{k_B T^2} \int \frac{d^2k}{(2\pi)^2}\, k^4\, \frac{e^{Jk^2/k_B T}}{\big(e^{Jk^2/k_B T} - 1\big)^2} \\ &= \frac{\tau k_B^2 T}{\pi \hbar^2} \int_0^\infty ds\, \frac{s^2\, e^s}{\big(e^s - 1\big)^2} = \frac{\zeta(2)}{\pi}\, \frac{\tau k_B^2 T}{\hbar^2} \quad .\end{aligned}$$

Here we have used

$$\int_0^\infty ds\, \frac{s^\alpha\, e^s}{\big(e^s - 1\big)^2} = \int_0^\infty ds\, \frac{\alpha\, s^{\alpha-1}}{e^s - 1} = \Gamma(\alpha+1)\, \zeta(\alpha) \quad .$$

(c) The energy current is

$$j^\mu_\varepsilon = \frac{2J}{\hbar} \int \frac{d^2k}{(2\pi)^2}\, Jk^2\, k^\mu\, \delta f(\boldsymbol{k}) = -\kappa\, \frac{\partial T}{\partial x^\mu} \quad .$$

We therefore repeat the calculation from part (c), including an extra factor of Jk^2 inside the integral. Thus,

$$\begin{aligned}\kappa &= \frac{2\tau}{\hbar^2}\,\frac{J^4}{k_{\rm B}T^2}\int\frac{d^2k}{(2\pi)^2}\,k^6\,\frac{e^{Jk^2/k_{\rm B}T}}{\big(e^{Jk^2/k_{\rm B}T}-1\big)^2}\\ &= \frac{\tau k_{\rm B}^3T^2}{\pi\hbar^2}\int\limits_0^\infty ds\,\frac{s^3\,e^s}{\big(e^s-1\big)^2} = \frac{6\,\zeta(3)}{\pi}\,\frac{\tau k_{\rm B}^3T^2}{\hbar^2}\quad .\end{aligned}$$

(8.8) Consider the collisionless Boltzmann equation for the Hamiltonian $\hat{H}(p) = \frac{1}{4}Ap^4$ in one space dimension. Suppose the initial distribution is given by

$$f(x,p,t=0) = C\, e^{-x^2/2\sigma^2}\, e^{-p^2/2\kappa^2} \quad .$$

(a) Find $f(x,p,t)$ for all $t > 0$.

(b) Find the equation for the locus of points (x,p) for which $f(x,p,t) = \exp(-\alpha^2/2)$.

(c) Express your result in (b) in dimensionless form and plot it for various values of the dimensionless time.

Solution :

(a) The velocity is $v(p) = \partial\varepsilon/\partial p = Ap^3$ and thus

$$f(x,p,t) = f\big(x - v(p)t, p, 0\big) = C\, e^{-(x-Ap^3t)^2/2\sigma^2}\, e^{-p^2/2\kappa^2} \quad .$$

(b) Clearly we have

$$\frac{(x-Ap^3t)^2}{\sigma^2} + \frac{p^2}{\kappa^2} = \alpha^2 \quad \Rightarrow \quad x(p,t;\alpha) = Ap^3t \pm \sigma\sqrt{\alpha^2 - \frac{p^2}{\kappa^2}} \quad .$$

If we write

$$\bar{x} \equiv \frac{x}{\sigma} \quad , \quad \bar{p} \equiv \frac{p}{\kappa} \quad , \quad s \equiv \frac{A\kappa^3\, t}{\sigma} \quad ,$$

then in dimensionless form we have

$$\bar{x}(\bar{p},s;\alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2} \quad .$$

(c) See fig. 1.

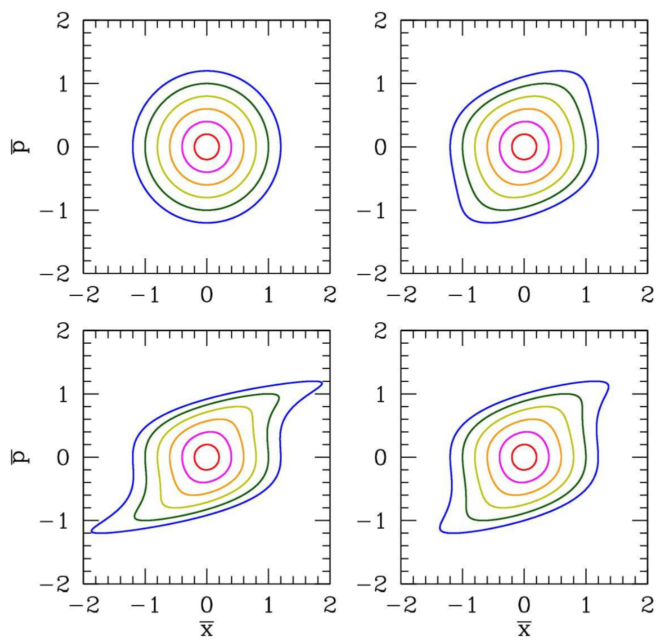

Figure 1: Level sets $\bar{x}(\bar{p},s;\alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2}$ for $\alpha = 0.2$ (red) to $\alpha = 1.2$ (blue) and $s = 0$ (upper left), $s = \frac{1}{3}$ (upper right), $s = \frac{2}{3}$ (lower right), and $s = 1$ (lower left). Compare with fig. 8.1 of the lecture notes.

(8.9) Consider an ideal gas of point particles in $d = 3$ dimensions with isotropic dispersion $\varepsilon(p) = Ap^\sigma$.

(a) Find the enthalpy per particle $h = \mu + Ts$, where μ is the chemical potential and s is the entropy per particle. (You may find it useful to review some of the material in chapter 4 of the notes.)

(b) Find the thermal conductivity κ within the relaxation time approximation.

Solution :

(a) For the dispersion $\varepsilon(p) = Ap^\sigma$, in d dimensions, one obtains the density of states $g(\varepsilon) = C\varepsilon^{r-1}$ where $r = d/\sigma$ and C is a constant. One then obtains the Helmholtz free energy

$$\begin{aligned} F &= -Nk_\mathrm{B}T\log(V/N) - Nk_\mathrm{B}T - Nk_\mathrm{B}T\log\int_0^\infty d\varepsilon\, g(\varepsilon)\, \exp(-\varepsilon/k_\mathrm{B}T) \\ &= -Nk_\mathrm{B}T\log(V/N) - Nk_\mathrm{B}T\log\big[eA\,\Gamma(r)\big] - rNk_\mathrm{B}T\log(k_\mathrm{B}T) \quad . \end{aligned}$$

We then have

$$\begin{aligned} \mu &= \left(\frac{\partial F}{\partial N}\right)_{T,V} = -k_\mathrm{B}T\log(V/N) - k_\mathrm{B}T\log\big[A\,\Gamma(r)\big] - rk_\mathrm{B}T\log(k_\mathrm{B}T) \\ s &= -\frac{1}{N}\left(\frac{\partial F}{\partial T}\right)_{V.N} = k_\mathrm{B}\log(V/N) + k_\mathrm{B}\log\big[eA\,\Gamma(r)\big] + rk_\mathrm{B}\log(ek_\mathrm{B}T) \quad , \end{aligned}$$

and thus

$$h = \mu + Ts = (r+1)\,k_\mathrm{B} \quad .$$

The thermal conductivity is obtained from the energy current $\boldsymbol{j}_\varepsilon = -\kappa\,\boldsymbol{\nabla}T$, where

$$j^\alpha_\varepsilon = \int d^dp\,\varepsilon\, v^\alpha\,\delta f = -\frac{n\tau}{k_\mathrm{B}T^2}\,\big\langle v^\alpha v^\beta\,\varepsilon\,(\varepsilon - h)\big\rangle\,\frac{\partial T}{\partial x^\beta} \quad ,$$

where we have used the result (see eqn. 8.71 of the lecture notes)

$$\delta f = -\frac{\tau}{k_\mathrm{B}T^2}\,(\varepsilon - h)(\boldsymbol{v}\cdot\boldsymbol{\nabla}T)\,f^0$$

and $h = c_pT$ for our ideal gas. Thus from isotropy we can replace $\langle v^\alpha v^\beta \to d^{-1}\boldsymbol{v}^2\,\delta^{\alpha\beta}$, in which case

$$\kappa = \frac{n\tau}{dk_\mathrm{B}T^2}\,\big\langle \boldsymbol{v}^2\,\varepsilon\,(\varepsilon - h)\big\rangle \quad .$$

With $\varepsilon(p) = Ap^\sigma$ we have $\boldsymbol{v}(\boldsymbol{p}) = \sigma Ap^{\sigma-1}\hat{\boldsymbol{p}}$ and thus $\boldsymbol{v}^2 = \sigma^2 A^{2/\sigma}\,\varepsilon^{2-2\sigma^{-1}}$. We then have

$$\kappa = \frac{n\tau}{dk_\mathrm{B}T^2}\,\sigma^2 A^{2/\sigma}\big\langle\varepsilon^{3-2\sigma^{-1}}\,\big(\varepsilon - (r+1)\,k_\mathrm{B}T\big)\big\rangle$$

Now with the density of states $g(\varepsilon) \propto \varepsilon^{r-1}$ we have

$$\langle\varepsilon^s\rangle = \int_0^\infty d\varepsilon\,\varepsilon^{r-1+s}\,e^{-\beta\varepsilon} \bigg/ \int_0^\infty d\varepsilon\,\varepsilon^{r-1}\,e^{-\beta\varepsilon} = \frac{\Gamma(r+s)}{\Gamma(r)}\,(k_\mathrm{B}T)^s \quad .$$

Working out the remaining details, we arrive at the expression

$$\kappa = \frac{2\sigma(\sigma-1)\,\Gamma(r+3-2\sigma^{-1})}{d\,\Gamma(r)}\,A^{2/\sigma}\cdot n\tau k_\mathrm{B}\cdot(k_\mathrm{B}T)^{2(1-\sigma^{-1})} \quad .$$

As a check, if we set $\sigma = 2$ and $A = 1/2m$ (ballistic dispersion), then for $d = 3$ we have $r = \frac{3}{2}$ and find $\kappa = \frac{5}{2}\,n\tau k_\mathrm{B}^2 T$, which is identical to the result in §8.4.4 of the lecture notes.