PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #6 SOLUTIONS

(1) Consider the collisionless Boltzmann equation for the Hamiltonian $\hat{H}(p) = \frac{1}{4}Ap^4$ in one space dimension. Suppose the initial distribution is given by

$$f(x,p,t=0) = C\,e^{-x^2/2\sigma^2}\,e^{-p^2/2\kappa^2} \quad .$$

(a) Find $f(x,p,t)$ for all $t > 0$.

(b) Find the equation for the locus of points (x,p) for which $f(x,p,t) = \exp(-\alpha^2/2)$.

(c) Express your result in (b) in dimensionless form and plot it for various values of the dimensionless time.

Solution :

(a) The velocity is $v(p) = \partial\varepsilon/\partial p = Ap^3$ and thus

$$f(x,p,t) = f\big(x - v(p)t, p, 0\big) = C\,e^{-(x-Ap^3t)^2/2\sigma^2}\;e^{-p^2/2\kappa^2} \quad .$$

(b) Clearly we have

$$\frac{(x-Ap^3t)^2}{\sigma^2} + \frac{p^2}{\kappa^2} = \alpha^2 \quad ,$$

which may be written as

$$x(p,t;\alpha) = Ap^3t \pm \sigma\sqrt{\alpha^2 - \frac{p^2}{\kappa^2}} \quad .$$

If we write

$$\bar{x} \equiv \frac{x}{\sigma} \quad , \quad \bar{p} \equiv \frac{p}{\kappa} \quad , \quad s \equiv \frac{A\kappa^3\,t}{\sigma} \quad ,$$

then in dimensionless form we have

$$\bar{x}(\bar{p},s;\alpha) = \bar{p}^3 s \pm \sqrt{\alpha^2 - \bar{p}^2} \quad .$$

(c) See fig. 1.

(2) Consider an ideal gas of point particles in $d = 3$ dimensions with isotropic dispersion $\varepsilon(p) = Ap^\sigma$.

(a) Find the enthalpy per particle $h = \mu + Ts$, where μ is the chemical potential and s is the entropy per particle. (You may find it useful to review some of the material in chapter 4 of the notes.)

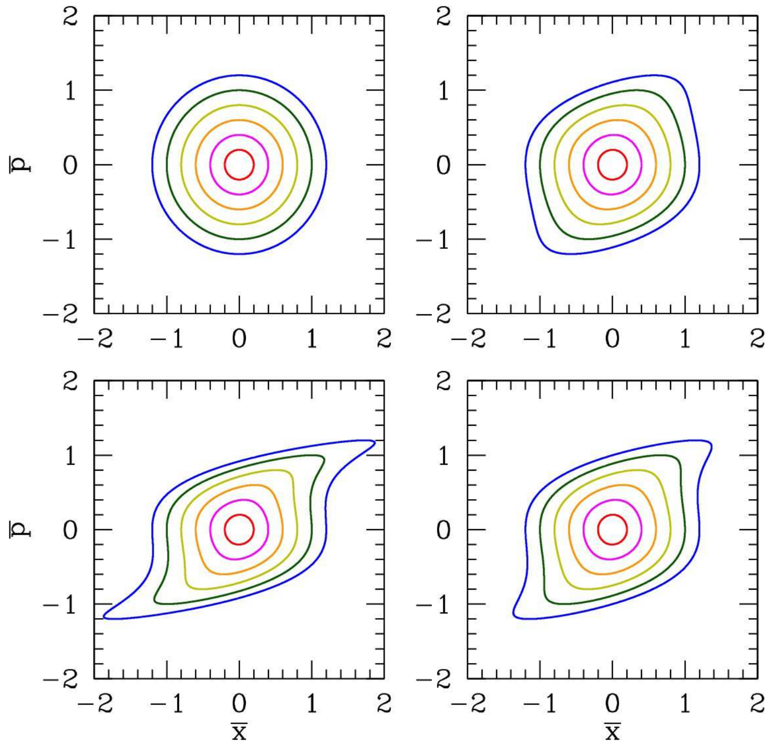

Figure 1: Level sets $\bar{x}(\bar{p},s;\alpha)=\bar{p}^3 s\pm\sqrt{\alpha^2-\bar{p}^2}$ for $\alpha=0.2$ (red) to $\alpha=1.2$ (blue) and $s=0$ (upper left), $s=\frac{1}{3}$ (upper right), $s=\frac{2}{3}$ (lower right), and $s=1$ (lower left). Compare with fig. 8.1 of the lecture notes.

(b) Find the thermal conductivity κ within the relaxation time approximation.

Solution :

(a) For the dispersion $\varepsilon(p)=Ap^\sigma$, in d dimensions, one obtains the density of states $g(\varepsilon)=C\varepsilon^{r-1}$ where $r=d/\sigma$ and C is a constant. One then obtains the Helmholtz free energy

$$
\begin{aligned}
F &= -Nk_{\rm B}T\log(V/N)-Nk_{\rm B}T-Nk_{\rm B}T\log\int\limits_0^\infty d\varepsilon\; g(\varepsilon)\;\exp(-\varepsilon/k_{\rm B}T)\\
&= -Nk_{\rm B}T\log(V/N)-Nk_{\rm B}T\log\big[eA\,\Gamma(r)\big]-rNk_{\rm B}T\log(k_{\rm B}T)\quad .
\end{aligned}
$$

We then have

$$
\begin{aligned}
\mu &= \left(\frac{\partial F}{\partial N}\right)_{T,V} = -k_{\rm B}T\log(V/N)-k_{\rm B}T\log\big[A\,\Gamma(r)\big]-rk_{\rm B}T\log(k_{\rm B}T)\\
s &= -\frac{1}{N}\left(\frac{\partial F}{\partial T}\right)_{V.N} = k_{\rm B}\log(V/N)+k_{\rm B}\log\big[eA\,\Gamma(r)\big]+rk_{\rm B}\log(ek_{\rm B}T)\quad ,
\end{aligned}
$$

and thus

$$
h=\mu+Ts=(r+1)\,k_{\rm B}\quad .
$$

The thermal conductivity is obtained from the energy current $\boldsymbol{j}_\varepsilon = -\kappa\, \boldsymbol{\nabla} T$, where

$$j^\alpha_\varepsilon = \int d^d\!p \, \varepsilon \, v^\alpha \, \delta f = -\frac{n\tau}{k_B T^2} \big\langle v^\alpha v^\beta \, \varepsilon\, (\varepsilon - h) \big\rangle \frac{\partial T}{\partial x^\beta} \quad ,$$

where we have used the result (see eqn. 8.71 of the lecture notes)

$$\delta f = -\frac{\tau}{k_B T^2} \, (\varepsilon - h)(\boldsymbol{v} \cdot \boldsymbol{\nabla} T) \, f^0$$

and $h = c_p T$ for our ideal gas. Thus from isotropy we can replace $\big\langle v^\alpha v^\beta \to d^{-1} \boldsymbol{v}^2 \, \delta^{\alpha\beta}$, in which case

$$\kappa = \frac{n\tau}{d k_B T^2} \big\langle \boldsymbol{v}^2 \, \varepsilon \, (\varepsilon - h) \big\rangle \quad .$$

With $\varepsilon(p) = A p^\sigma$ we have $\boldsymbol{v}(\boldsymbol{p}) = \sigma A p^{\sigma - 1} \hat{\boldsymbol{p}}$ and thus $\boldsymbol{v}^2 = \sigma^2 A^{2/\sigma} \, \varepsilon^{2 - 2\sigma^{-1}}$. We then have

$$\kappa = \frac{n\tau}{d k_B T^2} \, \sigma^2 A^{2/\sigma} \big\langle \varepsilon^{3 - 2\sigma^{-1}} \big(\varepsilon - (r+1)\, k_B T \big) \big\rangle$$

Now with the density of states $g(\varepsilon) \propto \varepsilon^{r-1}$ we have

$$\langle \varepsilon^s \rangle = \int\limits_0^\infty \! d\varepsilon \; \varepsilon^{r-1+s} \, e^{-\beta\varepsilon} \bigg/ \int\limits_0^\infty \! d\varepsilon \; \varepsilon^{r-1} \, e^{-\beta\varepsilon} = \frac{\Gamma(r+s)}{\Gamma(r)} \, (k_B T)^s \quad .$$

Working out the remaining details, we arrive at the expression

$$\kappa = \frac{2\sigma(\sigma - 1)\, \Gamma(r + 3 - 2\sigma^{-1})}{d\, \Gamma(r)} \, A^{2/\sigma} \cdot n\tau k_B \cdot (k_B T)^{2(1 - \sigma^{-1})} \quad .$$

As a check, if we set $\sigma = 2$ and $A = 1/2m$ (ballistic dispersion), then for $d = 3$ we have $r = \frac{3}{2}$ and find $\kappa = \frac{5}{2} \, n \tau k_B^2 T$, which is identical to the result in §8.4.4 of the lecture notes.