

PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #1 SOLUTIONS

(1) Consider a two-dimensional gas of fermions which obey the dispersion relation

$$\varepsilon(\mathbf{k}) = \varepsilon_0 \left((k_x^2 + k_y^2) a^2 + \frac{1}{2}(k_x^4 + k_y^4) a^4 \right) .$$

Sketch, on the same plot, the Fermi surfaces for $\varepsilon_F = 0.1 \varepsilon_0$, $\varepsilon_F = \varepsilon_0$, and $\varepsilon_F = 10 \varepsilon_0$.

SOLUTION : It is convenient to adimensionalize, writing

$$x \equiv k_x a \quad , \quad y \equiv k_y a \quad , \quad \nu \equiv \frac{\varepsilon}{\varepsilon_0} .$$

Then the equation for the Fermi surface becomes

$$x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4 = \nu .$$

In other words, we are interested in the *level sets* of the function $\nu(x, y) \equiv x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$. When ν is small, we can ignore the quartic terms, and we have an isotropic dispersion, with $\nu = x^2 + y^2$. *I.e.* we can write $x = \nu^{1/2} \cos \theta$ and $y = \nu^{1/2} \sin \theta$. The quartic terms give a contribution of order ν^4 , which is vanishingly small compared with the quadratic term in the $\nu \rightarrow 0$ limit. When $\nu \sim \mathcal{O}(1)$, the quadratic and quartic terms in the dispersion are of the same order of magnitude, and the continuous $O(2)$ symmetry, namely the symmetry under rotation by any angle, is replaced by a discrete symmetry group, which is the group of the square, known as C_{4v} in group theory parlance. This group has eight elements:

$$\{\mathbb{I} \quad , \quad R \quad , \quad R^2 \quad , \quad R^3 \quad , \quad \sigma \quad , \quad \sigma R \quad , \quad \sigma R^2 \quad , \quad \sigma R^3\}$$

Here R is the operation of counterclockwise rotation by 90° , sending (x, y) to $(-y, x)$, and σ is reflection in the y -axis, which sends (x, y) to $(-x, y)$. One can check that the function $\nu(x, y)$ is invariant under any of these eight operations from C_{4v} .

Explicitly, we can set $y = 0$ and solve the resulting quadratic equation in x^2 to obtain the maximum value of x , which we call $a(\nu)$. One finds

$$\frac{1}{2}x^4 + x^2 - \nu = 0 \quad \implies \quad a = \sqrt{\sqrt{1 + 2\nu} - 1} .$$

So long as $x \in \{-a, a\}$, we can solve for $y(x)$:

$$y(x) = \pm \sqrt{\sqrt{1 + 2\nu - 2x^2 - x^4} - 1} .$$

A sketch of the level sets, showing the evolution from an isotropic (*i.e.* circular) Fermi surface at small ν , to surfaces with discrete symmetries, is shown in fig. 1.

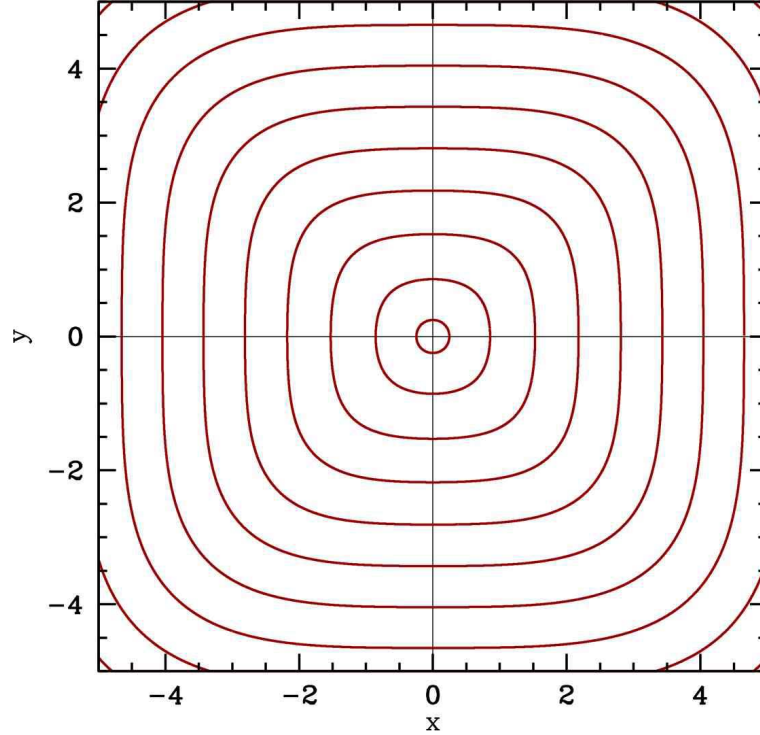


Figure 1: Level sets of the function $\nu(x, y) = x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$ for $\nu = (\frac{1}{2}n)^4$, with positive integer n .

(2) Using the Sommerfeld expansion, compute the heat capacity for a two-dimensional electron gas, to lowest nontrivial order in the temperature T .

SOLUTION : In the notes, in section 4.7.6, we obtained the result

$$\frac{E}{V} = \int_{-\infty}^{\varepsilon_F} d\varepsilon g(\varepsilon) \varepsilon + \frac{\pi^2}{6} (k_B T)^2 g(\varepsilon_F) + \mathcal{O}(T^4) \quad .$$

This entails a heat capacity of $C_{V,N} = V \cdot \frac{1}{3} \pi^2 k_B g(\varepsilon_F) \cdot k_B T$. The density of states at the Fermi level, $g(\varepsilon_F)$, is easily found to be

$$g(\varepsilon_F) = \frac{d}{2} \cdot \frac{n}{\varepsilon_F} \quad .$$

Thus,

$$C_{V,N} = N \cdot \frac{d \pi^2}{6} k_B \cdot \left(\frac{k_B T}{\varepsilon_F} \right) \quad ,$$

a form which is valid in any spatial dimension d .

(3) ${}^3\text{He}$ atoms consist of an odd number of fermions (two electrons, two protons, and one neutron), and hence is itself a fermion. Consider a kilomole of ${}^3\text{He}$ atoms at standard temperature and pressure ($T = 293$, K, $p = 1$ atm).

- (a) What is the Fermi temperature of the gas?
- (b) Calculate $\mu/k_B T$ and $\exp(-\mu/k_B T)$.
- (c) Find the average occupancy $n(\varepsilon)$ of a single particle state with energy $\frac{3}{2}k_B T$.

SOLUTION : Assuming the gas is essentially classical (this will be justified shortly), we find the gas density using the ideal gas law:

$$n = \frac{p}{k_B T} = \frac{1.013 \times 10^5 \text{ Pa}}{(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K})} = 2.51 \times 10^{25} \text{ m}^{-3} .$$

It is convenient to compute the rest energy of a ${}^3\text{He}$ atom. The mass is 3.016 amu (look it up on Google), hence

$$m_3 c^2 = 3.016 \cdot (931.5 \text{ MeV}) = 2.809 \text{ GeV} .$$

For the conversion of amu to MeV/c^2 , again try googling. We'll then need $\hbar c = 1973 \text{ eV} \cdot \text{\AA}$. (I remember 1973 because that was the summer I won third prize in an archery contest at Camp Mahakeno.) Thus,

$$\begin{aligned} \varepsilon_F &= \frac{(\hbar c)^2}{2m_3 c^2} \cdot (3\pi^2 n)^{2/3} = \frac{(1973 \text{ eV} \cdot \text{\AA})^2}{2.809 \times 10^9 \text{ eV}} \cdot (3\pi^2 \cdot 2.51 \times 10^{25} \text{ m}^{-3})^{2/3} \\ &= 1.14 \times 10^{-5} \text{ eV} . \end{aligned}$$

Now with $k_B = 86.2 \mu\text{eV}/\text{K}$, we have $T_F = \varepsilon_F/k_B = 0.13 \text{ K}$.

Within the GCE, the fugacity is given by $z = n\lambda_T^3$. The thermal wavelength is

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{1/2} = \left(\frac{2\pi \cdot (1973 \text{ eV} \cdot \text{\AA})^2}{(2.809 \times 10^9 \text{ eV}) \cdot (86.2 \times 10^{-6} \text{ eV/K}) \cdot (293 \text{ K})} \right)^{1/2} = 0.587 \text{ \AA} ,$$

hence

$$z = n\lambda_T^3 = (2.51 \times 10^{25} \text{ \AA}^{-3}) \cdot (0.587 \text{ \AA})^3 = 5.08 \times 10^{-6} .$$

Thus,

$$\frac{\mu}{k_B T} = \ln z = -12.2 , \quad e^{-\mu/k_B T} = z^{-1} = 1.97 \times 10^5 .$$

To find the occupancy $f(\varepsilon - \mu)$, we note $\varepsilon - \mu = \left[\frac{3}{2} - (-12.2) \right] k_B T = 13.7 k_B T$, in which case

$$n(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/k_B T} + 1} = \frac{1}{e^{13.7} + 1} = 1.12 \times 10^{-6} .$$

(4) For ideal Fermi gases in $d = 1, 2,$ and 3 dimensions, compute at $T = 0$ the average energy per particle E/N in terms of the Fermi energy ε_F .

SOLUTION : The number of particles is

$$N = g V \int \frac{d^d k}{(2\pi)^d} \Theta(k_F - k) = V \cdot \frac{g \Omega_d}{(2\pi)^d} \frac{k_F^d}{d} ,$$

where g is the internal degeneracy and Ω_d is the surface area of a sphere in d dimensions. The total energy is

$$E = g V \int \frac{d^d k}{(2\pi)^d} \frac{\hbar^2 k^2}{2m} \Theta(k_F - k) = V \cdot \frac{g \Omega_d}{(2\pi)^d} \frac{k_F^d}{d+2} \cdot \frac{\hbar^2 k_F^2}{2m} .$$

Therefore,

$$\frac{E}{N} = \frac{d}{d+2} \varepsilon_F .$$

(5) Obtain numerical estimates for the Fermi energy (in eV) and the Fermi temperature (in Kelvin) for the following systems:

- (a) conduction electrons in silver, lead, and aluminum
- (b) nucleons in a heavy nucleus, such as ^{200}Hg

SOLUTION : The Fermi energy for ballistic dispersion is given by

$$\varepsilon_F = \frac{\hbar^2}{2m^*} (3\pi^2 n)^{2/3} ,$$

where m^* is the effective mass, which one can assume is the electron mass $m = 9.11 \times 10^{-28}$ g. The electron density is given by the number of valence electrons of the atom divided by the volume of the unit cell. A typical unit cell volume is on the order of 30 \AA^3 , and if we assume one valence electron per atom we obtain a Fermi energy of $\varepsilon_F = 3.8 \text{ eV}$, and hence a Fermi temperature of $3.8 \text{ eV} / (86.2 \times 10^{-6} \text{ eV/K}) = 4.4 \times 10^4 \text{ K}$. This sets the overall scale. For detailed numbers, one can examine table 2.1 in *Solid State Physics* by Ashcroft and Mermin. One finds

$$T_F(\text{Ag}) = 6.38 \times 10^4 \text{ K} \quad ; \quad T_F(\text{Pb}) = 11.0 \times 10^4 \text{ K} \quad ; \quad T_F(\text{Al}) = 13.6 \times 10^4 \text{ K} .$$

Nuclear densities are of course much higher. In the literature one finds the relation $R \sim A^{1/3} r_0$, where R is the nuclear radius, A is the number of nucleons (*i.e.* the atomic mass number), and $r_0 \simeq 1.2 \text{ fm} = 1.2 \times 10^{-15} \text{ m}$. Under these conditions, the nuclear density is on the order of $n \sim 3A/4\pi R^3 = 3/4\pi r_0^3 = 1.4 \times 10^{44} \text{ m}^{-3}$. With the mass of the proton $m_p = 938 \text{ MeV}/c^2$ we find $\varepsilon_F \sim 30 \text{ MeV}$ for the nucleus, corresponding to a temperature of roughly $T_F \sim 3.5 \times 10^{11} \text{ K}$.

(6) Show that the chemical potential of a three-dimensional ideal nonrelativistic Fermi gas is given by

$$\mu(n, T) = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F} \right)^2 - \frac{\pi^4}{80} \left(\frac{k_B T}{\varepsilon_F} \right)^4 + \dots \right]$$

and the average energy per particle is

$$\frac{E}{N} = \frac{3}{5} \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\varepsilon_F} \right)^2 - \frac{\pi^4}{16} \left(\frac{k_B T}{\varepsilon_F} \right)^4 + \dots \right] ,$$

where $\mu_0(n)$ is the Fermi energy at $T = 0$. Compute the heat capacity $C_V(T)$ to terms of order T^3 . How does the T^3 contribution to the electronic heat capacity compare with the contribution from phonons?

SOLUTION : From the Sommerfeld expansion we have

$$\begin{aligned} \int_{-\infty}^{\infty} d\varepsilon \phi(\varepsilon) f(\varepsilon - \mu) &= \int_{-\infty}^{\mu} d\varepsilon \phi(\varepsilon) + \frac{\pi^2}{6} (k_B T)^2 \phi'(\mu) + \frac{7\pi^4}{360} (k_B T)^4 \phi'''(\mu) + \mathcal{O}(T^6) \\ &= \left\{ 1 + \frac{\pi^2}{6} (k_B T)^2 \frac{d^2}{d\mu^2} + \frac{7\pi^4}{360} (k_B T)^2 \frac{d^4}{d\mu^4} + \mathcal{O}(T^6) \right\} H(\mu) \quad , \end{aligned}$$

where $\phi(\varepsilon) = H'(\varepsilon)$. Let's work this out to second order in T^2 for the case $\phi(\varepsilon) = g(\varepsilon)$. The integral then gives the overall density n . We write $\mu = \varepsilon_F + \delta\mu$ and expand the RHS to second order in $\delta\mu$. Thus yields

$$\begin{aligned} n &= \int_{-\infty}^{\varepsilon_F} d\varepsilon g(\varepsilon) + g(\varepsilon_F) \delta\mu + \frac{1}{2} g'(\varepsilon_F) (\delta\mu)^2 + \dots \\ &\quad + \frac{\pi^2}{6} (k_B T)^2 g'(\varepsilon_F) + \frac{\pi^2}{6} (k_B T)^2 g''(\varepsilon_F) \delta\mu + \dots \\ &\quad + \frac{7\pi^4}{360} (k_B T)^4 g'''(\varepsilon_F) + \dots \quad . \end{aligned}$$

RHS of the first line above comes from expanding the integral in the first term on the RHS of the previous equation to second order in $\delta\mu$. The subsequent lines come from the expansions of the second and third terms on the RHS of the previous equation, respectively. We expand out to the necessary order in each case. From this equation we thus obtain

$$\delta\mu = -\frac{\pi^2}{6} (k_B T)^2 \frac{g'(\varepsilon_F)}{g(\varepsilon_F)} - \frac{\pi^4}{36} (k_B T)^4 \cdot \left[\frac{1}{2} \left(\frac{g'(\varepsilon_F)}{g(\varepsilon_F)} \right)^3 - \frac{g'(\varepsilon_F) g''(\varepsilon_F)}{g^2(\varepsilon_F)} + \frac{7}{10} \frac{g'''(\varepsilon_F)}{g(\varepsilon_F)} \right] + \dots$$

If we assume that $g(\varepsilon)$ is a homogeneous function with $g(\varepsilon) \propto \varepsilon^\alpha$, then find

$$\delta\mu = -\frac{\alpha \pi^2}{6} \frac{(k_B T)^2}{\varepsilon_F} - \alpha(\alpha - 2)(2\alpha - 7) \frac{\pi^4}{360} \frac{(k_B T)^4}{\varepsilon_F^3} + \mathcal{O}(T^6) \quad .$$

Substituting $\alpha = \frac{1}{2}$, as is appropriate for three-dimensional ballistic fermions, we obtain

$$\delta\mu = -\frac{\pi^2}{12} \frac{(k_B T)^2}{\varepsilon_F} - \frac{\pi^4}{80} \frac{(k_B T)^4}{\varepsilon_F^3} + \mathcal{O}(T^6) \quad ,$$

which is the desired result.

The result for the energy is tedious to carry out by hand, but is rather straightforward using a symbolic manipulation program such as Mathematica or Maple. We assume the density of states is of the form $g(\varepsilon) = C\varepsilon^\alpha$. Then from the Sommerfeld expansion we have

$$\begin{aligned} \frac{E}{V} &= \frac{C\mu^{\alpha+2}}{\alpha+2} \left\{ 1 + \alpha(\alpha+1) \frac{\pi^2}{6} \left(\frac{k_B T}{\mu} \right)^2 + (\alpha-2)(\alpha-1)\alpha(\alpha+1) \frac{7\pi^4}{360} \left(\frac{k_B T}{\mu} \right)^4 + \dots \right\} \\ \frac{N}{V} &= \frac{C\mu^{\alpha+1}}{\alpha+2} \left\{ 1 + (\alpha-1)\alpha \frac{\pi^2}{6} \left(\frac{k_B T}{\mu} \right)^2 + (\alpha-3)(\alpha-2)(\alpha-1)\alpha \frac{7\pi^4}{360} \left(\frac{k_B T}{\mu} \right)^4 + \dots \right\} \end{aligned}$$

Carefully taking the ratio and evaluating to order T^4 , we find

$$\frac{E}{N} = \left(\frac{\alpha+1}{\alpha+2} \right) \mu \cdot \left\{ 1 + (\alpha+1) \frac{\pi^2}{3} \left(\frac{k_B T}{\mu} \right)^2 + \alpha(\alpha+1)(\alpha-6) \frac{\pi^4}{45} \left(\frac{k_B T}{\mu} \right)^4 + \dots \right\} \quad .$$

Unfortunately we're not quite done, since we now must expand μ in a power series in T , invoking our previous result. Working this out (by hand!), I obtain

$$\frac{E}{N} = \left(\frac{\alpha+1}{\alpha+2} \right) \varepsilon_F \cdot \left\{ 1 + (\alpha+2) \frac{\pi^2}{6} \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \alpha(\alpha+2)(2\alpha-7) \frac{\pi^4}{120} \left(\frac{k_B T}{\varepsilon_F} \right)^4 + \mathcal{O}(T^6) \right\} \quad .$$

Setting $\alpha = \frac{1}{2}$ we have $\frac{1}{6}(\alpha+2) = \frac{5}{12}$ and $\frac{1}{120}\alpha(\alpha+2)(2\alpha-7) = -\frac{1}{16}$, as indicated in the statement of the problem. Our formula holds for general α , so we can find the result for $d = 2$ by setting $\alpha = 0$.

The heat capacity is

$$\begin{aligned} C_{V,N} &= \left(\frac{\partial E}{\partial T} \right)_{V,N} \\ &= Nk_B \left\{ (\alpha+1) \frac{\pi^2}{3} \left(\frac{k_B T}{\varepsilon_F} \right) + \alpha(\alpha+1)(2\alpha-7) \frac{\pi^4}{30} \left(\frac{k_B T}{\varepsilon_F} \right)^3 + \mathcal{O}(T^5) \right\} \quad . \end{aligned}$$

In $d = 3$, with $\alpha = \frac{1}{2}$, the order T^3 term is $-\frac{3}{20}\pi^4(k_B T/\varepsilon_F)^3 Nk_B$. The low temperature phonon contribution is $C_V^{(\text{phonon})} = \frac{12}{5}\pi^4 (T/\Theta_D)^3 Nk_B$, where Θ_D is the Debye temperature. The ratio is

$$\frac{\Delta C_V^{(\text{elec})}}{C_V^{(\text{phonon})}} = -\frac{1}{16} \left(\frac{\Theta_D}{T_F} \right)^3 \quad .$$

Since Θ_D is typically hundreds of K while T_F is tens of thousands of K, this ratio is on the order of 10^{-7} .