PHYSICS 140B : STATISTICAL PHYSICS
MIDTERM SOLUTIONS

(1) A one-dimensional spin chain is described by the Hamiltonian

$$\hat{H} = -\sum_n \Big(J S_n S_{n+1} + K S_n^2 S_{n+1}^2 \Big) \quad ,$$

where $S_n \in \{-1, 0, +1\}$ on each site. Find the transfer matrix. `[50 points]`

Solution:

The transfer matrix is

$$T_{SS'} = \exp\!\big(\beta J S S' + \beta K S^2 S'^2\big) = \begin{pmatrix} e^{\beta(K+J)} & 1 & e^{\beta(K-J)} \\ 1 & 1 & 1 \\ e^{\beta(K-J)} & 1 & e^{\beta(K+J)} \end{pmatrix}_{SS'} \quad ,$$

where the rows and columns are labeled by the spin polarizations $\{+1, 0, -1\}$.

(2) Consider a gas of ballistic particles in $d = 2$ dimensions with two-body interactions

$$u(r) = \frac{A}{r^3} \quad .$$

(a) For the cluster $\gamma = \bullet\!\!-\!\!\!-\!\!\bullet$, express b_γ as an integral over the radial coordinate and show that the expression is integrable both as $r \to 0$ and as $r \to \infty$. `[20 points]`

(b) Compute the second virial coefficient $B_2(T)$. Express any integral expressions as dimensionless integrals with dimensionful prefactors. You may find the following useful:

$$\int\limits_0^\infty ds\, \frac{1 - e^{-s}}{s^{5/3}} = -\Gamma(-\tfrac{2}{3}) = 4.01841\ldots \quad .$$

`[30 points]`

Solution:

(a) With $\gamma = \bullet\!\!-\!\!\!-\!\!\bullet$, we have

$$f(r) = \tfrac{1}{2} \int \frac{d^2 r}{\lambda_T^2}\, f(r) = \frac{\pi}{\lambda_T^2} \int\limits_0^\infty dr\, r \left[e^{-\beta A / r^3} - 1 \right] \quad .$$

As $r \to 0$, the Mayer function tends to $f(0) = -1$ and $dr\, r\, f(r) \approx -dr\, r$, which is integrable down to $r = 0$. As $r \to \infty$, we have $f(r) = -\beta A\, r^{-3} + \mathcal{O}(r^{-6})$, and thus $dr\, r\, f(r) \approx \beta A\, r^{-2} dr$ which is again integrable out to $r = \infty$.

(b) Taking the gigantic hint from the integral provided, substitute $s \equiv \beta A\, r^{-3}$, hence

$$r = (\beta A)^{1/3} s^{-1/3} \qquad \Rightarrow \qquad dr\, r\, f(r) = -\tfrac{1}{3} (\beta A)^{2/3} s^{-5/3} ds$$

and

$$b_\gamma(T) = \frac{\pi}{3}\left(\frac{A}{k_{\mathrm{B}}T}\right)^{2/3} \int\limits_0^\infty ds\, \frac{e^{-s}-1}{s^{5/3}} = \frac{\pi}{3}\,\Gamma(-\tfrac{2}{3})\left(\frac{A}{k_{\mathrm{B}}T}\right)^{2/3}$$

The second virial coefficient is then

$$B_2(T) = -b_\gamma(t) = -\frac{\pi}{3}\,\Gamma(-\tfrac{2}{3})\left(\frac{A}{k_{\mathrm{B}}T}\right)^{2/3} \simeq 4.21\left(\frac{A}{k_{\mathrm{B}}T}\right)^{2/3} \quad .$$