PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #4 SOLUTIONS

(1) Consider the equation of state

$$p\sqrt{v^2 - b^2} = RT\,\exp\!\left(-\frac{a}{RTv^2}\right).$$

(a) Find the critical point (v_c, T_c, p_c).

(b) Defining $\bar{p} = p/p_c$, $\bar{v} = v/v_c$, and $\bar{T} = T/T_c$, write the equation of state in dimensionless form $\bar{p} = \bar{p}(\bar{v}, \bar{T})$.

(c) Expanding $\bar{p} = 1 + \pi$, $\bar{v} = 1 + \epsilon$, and $\bar{T} = 1 + t$, find $\epsilon_{\rm liq}(t)$ and $\epsilon_{\rm gas}(t)$ for $-1 \ll t < 0$.

(2) Consider an Ising ferromagnet where the nearest neighbor exchange temperature is $J_{\rm NN}/k_B = 50\,{\rm K}$ and the next nearest neighbor exchange temperature is $J_{\rm NNN}/k_B = 10\,{\rm K}$. What is the mean field transition temperature T_c if the lattice is:

(a) square
(b) honeycomb
(c) triangular
(d) simple cubic
(e) body centered cubic

Hint : As an intermediate step, you might want to show that the mean field transition temperature is given by

$$k_B T_c^{\rm MF} = z_1\, J_{\rm NN} + z_2\, J_{\rm NNN},$$

where z_1 and z_2 are the number of nearest neighbors and next-nearest neighbors of a given lattice site, respectively.

(3) Consider a three state Ising model,

$$\hat{H} = -J\sum_{\langle ij\rangle} S_i\, S_j - B\sum_i S_i,$$

where $S_i \in \{-1\,,\,0\,,\,+1\}$.

(a) Writing $S_i = m + \delta S_i$ and ignoring terms quadratic in the fluctuations, derive the mean field Hamiltonian $H_{\rm MF}$.

(b) Find the dimensionless mean field free energy density, $f = F_{\rm MF}/NzJ$, where z is the lattice coordination number. You should define the dimensionless temperature $\theta \equiv k_B T/zJ$ and the dimensionless field $h \equiv B/zJ$.

(c) Find the self-consistency equation for $m = \langle S_i\rangle$ and show that this agrees with the condition $\partial f/\partial m = 0$.

(d) Expand $f(m)$ to fourth order in m and first order in h.

(e) Find the critical temperature θ_c.

(f) Find $m(\theta_\text{c}, h)$.

(4) For the O(3) Heisenberg ferromagnet,

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\boldsymbol{\Omega}}_i \cdot \hat{\boldsymbol{\Omega}}_j \ ,$$

find the mean field transition temperature T_c^MF. Here, each $\hat{\boldsymbol{\Omega}}_i$ is a three-dimensional unit vector, which can be parameterized using the usual polar and azimuthal angles:

$$\hat{\boldsymbol{\Omega}}_i = \big(\sin\theta_i \, \cos\phi_i \, , \, \sin\theta_i \, \sin\phi_i \, , \, \cos\theta_i \big) \ .$$

The thermodynamic trace is defined as

$$\text{Tr}\, A(\hat{\boldsymbol{\Omega}}_1 \, , \, \ldots \, , \, \hat{\boldsymbol{\Omega}}_N) = \int \prod_{i=1}^N \frac{d\Omega_i}{4\pi} \, A(\hat{\boldsymbol{\Omega}}_1 \, , \, \ldots \, , \, \hat{\boldsymbol{\Omega}}_N) \ ,$$

where

$$d\Omega_i = \sin\theta_i \, d\theta_i \, d\phi_i \ .$$

Hint : Your mean field Ansatz will look like $\hat{\boldsymbol{\Omega}}_i = \boldsymbol{m} + \delta\boldsymbol{\Omega}_i$*, where* $\boldsymbol{m} = \langle \boldsymbol{\Omega}_i \rangle$*. You'll want to ignore terms in the Hamiltonian which are quadratic in fluctuations, i.e.* $\delta\boldsymbol{\Omega}_i \cdot \delta\boldsymbol{\Omega}_j$*. You can, without loss of generality, assume* $\boldsymbol{m}$ *to lie in the* $\hat{\boldsymbol{z}}$ *direction.*