PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #2

(1) Consider a model in which there are three possible states per site, which we can denote by A, B, and V. The states A and B are for our purposes identical. The energies of A-A, A-B, and B-B links are all identical and equal to W. The state V represents a vacancy, and any link containing a vacancy, meaning A-V, B-V, or V-V, has energy 0.

(a) Suppose we write $\sigma = +1$ for A, $\sigma = -1$ for B, and $\sigma = 0$ for V. How would you write a Hamiltonian for this system? Your result should be of the form

$$\hat{H} = \sum_{\langle ij \rangle} E(\sigma_i \, , \, \sigma_j) \quad .$$

Find a simple and explicit function $E(\sigma, \sigma')$ which yields the correct energy for each possible bond configuration.

(b) Consider a triangle of three sites. Find the average total energy at temperature T. *There are $3^3 = 27$ states for the triangle. You can just enumerate them all and find the energies.*

(c) For a one-dimensional ring of N sites, find the 3×3 transfer matrix R. Find the free energy per site $F(T,N)/N$ and the ground state entropy per site $S(T,N)/N$ in the $N \to \infty$ limit for the cases $W < 0$ and $W > 0$. Interpret your results. *The eigenvalue equation for R factorizes, so you only have to solve a quadratic equation.*

(2) Consider a longer-ranged Ising model on a ring with Hamiltonian

$$\hat{H} = \sum_{n=1}^{L} E(\sigma_n, \sigma_{n+1}, \sigma_{n+2}) \quad ,$$

where the length $L > 2$ is even and the energy function $E(\sigma.\sigma', \sigma'')$ is arbitrary. Note that this chain involves second-neighbor interactions.

(a) Show that the partition function can be written as

$$Z(T,L) = \mathop{\mathsf{Tr}}_{\boldsymbol{\sigma}} \prod_{k=1}^{K} T(\sigma_{2k-1}, \sigma_{2k} \,|\, \sigma_{2k+1}, \sigma_{2k+2})$$

where $K = \frac{1}{2}L$. and find $T(\sigma_{2k-1}, \sigma_{2k} \,|\, \sigma_{2k+1}, \sigma_{2k+2})$ in terms of $E(\sigma_n, \sigma_{n+1}, \sigma_{n+2})$.

(b) Show that $T(\sigma, \sigma' \,|\, \sigma'', \sigma''')$ can be considered a 4×4 matrix, whence $Z = \mathsf{Tr}\, T^K$.

(c) Suppose

$$E(\sigma_n, \sigma_{n+1}, \sigma_{n+2}) = \begin{cases} -J & \text{if } \sigma_n = \sigma_{n+1} = \sigma_{n+2} \\ 0 & \text{otherwise} \quad . \end{cases}$$

Find the 4×4 transfer matrix.

(3) The Blume-Capel model is a spin-1 version of the Ising model, with Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i \, S_j - \Delta \sum_i S_i^2 \ ,$$

where $S_i \in \{-1\,,\,0\,,\,+1\}$ and where the first sum is over all links of a lattice and the second sum is over all sites. It has been used to describe magnetic solids containing vacancies ($S = 0$ for a vacancy) as well as phase separation in ${}^4\mathrm{He} - {}^3\mathrm{He}$ mixtures ($S = 0$ for a ^{4}He atom). This problem will give you an opportunity to study and learn the material in §§5.2,3 of the notes. For parts (b), (c), and (d) you should work in the thermodynamic limit. The eigenvalues and eigenvectors are such that it would shorten your effort considerably to use a program like `Mathematica` to obtain them.

(a) Find the transfer matrix for the $d = 1$ Blume-Capel model.

(b) Find the free energy $F(T, \Delta, N)$.

(c) Find the density of $S = 0$ sites as a function of T and Δ.

(d) *Exciting!* Find the correlation function $\langle \, S_j \, S_{j+n} \, \rangle$.

(4) Consider an N-site Ising ring, with N even. Let $K = J/k_{\mathrm{B}}T$ be the dimensionless ferromagnetic coupling ($K > 0$), and $\mathcal{H}(K, N) = H/k_{\mathrm{B}}T = -K \sum_{n=1}^{N} \sigma_n \, \sigma_{n+1}$ the dimensionless Hamiltonian. The partition function is $Z(K, N) = \mathsf{Tr}\ e^{-\mathcal{H}(K,N)}$. By 'tracing out' over the even sites, show that

$$Z(K, N) = e^{-N'c} \, Z(K', N') \ ,$$

where $N' = N/2$, $c = c(K)$ and $K' = K'(K)$. Thus, the partition function of an N site ring with dimensionless coupling K is related to the partition function *for the same model* on an $N' = N/2$ site ring, at some *renormalized* coupling K', up to a constant factor.