9 Stochastic Processes : Worked Examples

(9.1) Due to quantum coherence effects in the backscattering from impurities, one-dimensional wires don't obey Ohm's law (in the limit where the 'inelastic mean free path' is greater than the sample dimensions, which you may assume). Rather, let $\mathcal{R}(L) = R(L)/(h/e^2)$ be the dimensionless resistance of a quantum wire of length L, in units of $h/e^2 = 25.813\,\mathrm{k}\Omega$. Then the dimensionless resistance of a quantum wire of length $L + \delta L$ is given by

$$\mathcal{R}(L+\delta L) = \mathcal{R}(L) + \mathcal{R}(\delta L) + 2\,\mathcal{R}(L)\,\mathcal{R}(\delta L) + 2\cos\alpha\,\sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]} \quad ,$$

where α is a *random phase* uniformly distributed over the interval $[0, 2\pi)$. Here,

$$\mathcal{R}(\delta L) = \frac{\delta L}{2\ell} \quad ,$$

is the dimensionless resistance of a small segment of wire, of length $\delta L \lesssim \ell$, where ℓ is the 'elastic mean free path'. (Using the Boltzmann equation, we would obtain $\ell = 2\pi\hbar n\tau/m$.)

Show that the distribution function $P(\mathcal{R}, L)$ for resistances of a quantum wire obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell}\,\frac{\partial}{\partial \mathcal{R}}\bigg\{\mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}}\bigg\} \quad .$$

Show that this equation* may be solved in the limits $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$, with

$$P(\mathcal{R}, z) = \frac{1}{z}\,e^{-\mathcal{R}/z}$$

for $\mathcal{R} \ll 1$, and

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\,\frac{1}{\mathcal{R}}\,e^{-(\ln\mathcal{R} - z)^2/4z}$$

for $\mathcal{R} \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle\mathcal{R}\rangle$ in the former case, and $\langle\ln\mathcal{R}\rangle$ in the latter case.

Solution :

From the composition rule for series quantum resistances, we derive the phase averages

$$\begin{aligned}
\big\langle\delta\mathcal{R}\big\rangle &= \Big(1 + 2\,\mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell} \\
\big\langle(\delta\mathcal{R})^2\big\rangle &= \Big(1 + 2\,\mathcal{R}(L)\Big)^2\,\bigg(\frac{\delta L}{2\ell}\bigg)^2 + 2\,\mathcal{R}(L)\,\Big(1 + \mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell}\,\bigg(1 + \frac{\delta L}{2\ell}\bigg) \\
&= 2\,\mathcal{R}(L)\,\Big(1 + \mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell} + \mathcal{O}\big((\delta L)^2\big) \quad ,
\end{aligned}$$

whence we obtain the drift and diffusion terms

$$F_1(\mathcal{R}) = \frac{2\,\mathcal{R} + 1}{2\ell} \qquad , \qquad F_2(\mathcal{R}) = \frac{2\mathcal{R}(1+\mathcal{R})}{2\ell} \quad .$$

Note that $2F_1(\mathcal{R}) = dF_2/d\mathcal{R}$, which allows us to write the Fokker-Planck equation as

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial \mathcal{R}}\bigg\{\frac{\mathcal{R}\,(1+\mathcal{R})}{2\ell}\,\frac{\partial P}{\partial \mathcal{R}}\bigg\} \quad .$$

Defining the dimensionless length $z = L/2\ell$, we have

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}}\bigg\{\mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}}\bigg\} \quad .$$

In the limit $\mathcal{R} \ll 1$, this reduces to

$$\frac{\partial P}{\partial z} = \mathcal{R}\,\frac{\partial^2 P}{\partial \mathcal{R}^2} + \frac{\partial P}{\partial \mathcal{R}} \quad ,$$

which is satisfied by $P(\mathcal{R}, z) = z^{-1}\,\exp(-\mathcal{R}/z)$. For this distribution one has $\langle \mathcal{R} \rangle = z$.

In the opposite limit, $\mathcal{R} \gg 1$, we have

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}}\left(\mathcal{R}^2 \frac{\partial}{\partial \mathcal{R}}\right) = \frac{\partial^2 P}{\partial \nu^2} + \frac{\partial P}{\partial \nu} \quad ,$$

where $\nu \equiv \ln \mathcal{R}$. This is solved by the log-normal distribution,

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\, e^{-(\nu+z)^2/4z} \quad .$$

Note that

$$P(\mathcal{R}, z)\, d\mathcal{R} = \widetilde{P}(\nu, z)\, d\nu = \frac{1}{\sqrt{4\pi z}}\, e^{-(\nu-z)^2/4z}\, d\nu \quad ,$$

One then obtains $\langle \nu \rangle = \langle \ln \mathcal{R} \rangle = z$. Furthermore,

$$\langle \mathcal{R}^n \rangle = \langle e^{n\nu} \rangle = \frac{1}{\sqrt{4\pi z}} \int\limits_{-\infty}^{\infty} d\nu\, e^{-(\nu-z)^2/4z}\, e^{n\nu} = e^{k(k+1)z}$$

Note then that $\langle \mathcal{R} \rangle = \exp(2z)$, so the mean resistance grows *exponentially* with length. However, note also that $\langle \mathcal{R}^2 \rangle = \exp(6z)$, so

$$\langle (\Delta \mathcal{R})^2 \rangle = \langle \mathcal{R}^2 \rangle - \langle \mathcal{R} \rangle^2 = e^{6z} - e^{4z} \quad ,$$

and so the standard deviation grows as $\sqrt{\langle \mathcal{R}^2 \rangle} \sim \exp(3z)$ which grows faster than $\langle \mathcal{R} \rangle$. In other words, the resistance $\mathcal{R}$ itself is not a *self-averaging* quantity, meaning the ratio of its standard deviation to its mean doesn't vanish in the thermodynamic limit – indeed it diverges. However, $\nu = \ln \mathcal{R}$ *is* a self-averaging quantity, with $\langle \nu \rangle = z$ and $\sqrt{\langle \nu^2 \rangle} = \sqrt{2z}$.

(9.2) Show that for time scales sufficiently greater than γ^{-1} that the solution $x(t)$ to the Langevin equation $\ddot{x}+\gamma\dot{x} = \eta(t)$ describes a Markov process. You will have to construct the matrix M defined in ch. 9 of the lecture notes. You should assume that the random force $\eta(t)$ is distributed as a Gaussian, with $\langle\eta(s)\rangle = 0$ and $\langle\eta(s)\,\eta(s')\rangle = \Gamma\,\delta(s-s')$.

Solution:

The probability distribution is

$$P(x_1,t_1\,;\,\ldots\,;\,x_N,t_N) = \text{det}^{-1/2}(2\pi M)\,\exp\left\{-\tfrac{1}{2}\sum_{j,j'=1}^{N} M^{-1}_{jj'}\,x_j\,x_{j'}\right\} \quad ,$$

where

$$M(t,t') = \int_0^t ds \int_0^{t'} ds'\, G(s-s')\,K(t-s)\,K(t'-s') \quad ,$$

and $K(s) = (1-e^{-\gamma s})/\gamma$. Thus,

$$\begin{aligned}
M(t,t') &= \frac{\Gamma}{\gamma^2}\int_0^{t_{\rm min}} ds\,(1-e^{-\gamma(t-s)})(1-e^{-\gamma(t'-s)}) \\
&= \frac{\Gamma}{\gamma^2}\left\{t_{\rm min} - \frac{1}{\gamma} + \frac{1}{\gamma}\left(e^{-\gamma t}+e^{-\gamma t'}\right) - \frac{1}{2\gamma}\left(e^{-\gamma|t-t'|}+e^{-\gamma(t+t')}\right)\right\} \quad .
\end{aligned}$$

In the limit where t and t' are both large compared to γ^{-1}, we have $M(t,t') = 2D\,\text{min}(t,t')$, where the diffusions constant is $D = \Gamma/2\gamma^2$. Thus,

$$M = 2D\begin{pmatrix}
t_1 & t_2 & t_3 & t_4 & t_5 & \cdots & t_N \\
t_2 & t_2 & t_3 & t_4 & t_5 & \cdots & t_N \\
t_3 & t_3 & t_3 & t_4 & t_5 & \cdots & t_N \\
t_4 & t_4 & t_4 & t_4 & t_5 & \cdots & t_N \\
t_5 & t_5 & t_5 & t_5 & t_5 & \cdots & t_N \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
t_N & t_N & t_N & t_N & t_N & \cdots & t_N
\end{pmatrix} \quad .$$

To find the determinant of M, subtract row 2 from row 1, then subtract row 3 from row 2, *etc.*The result is

$$\widetilde{M} = 2D\begin{pmatrix}
t_1-t_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
t_2-t_3 & t_2-t_3 & 0 & 0 & 0 & \cdots & 0 \\
t_3-t_4 & t_3-t_4 & t_3-t_4 & 0 & 0 & \cdots & 0 \\
t_4-t_5 & t_4-t_5 & t_4-t_5 & t_4-t_5 & 0 & \cdots & 0 \\
t_5-t_6 & t_5-t_6 & t_5-t_6 & t_5-t_6 & t_5-t_6 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
t_N & t_N & t_N & t_N & t_N & \cdots & t_N
\end{pmatrix} \quad .$$

Note that the last row is unchanged, since there is no row $N+1$ to subtract from it Since $\widetilde{M}$ is obtained from M by consecutive row additions, we have

$$\text{det}\,M = \text{det}\,\widetilde{M} = (2D)^N(t_1-t_2)(t_2-t_3)\cdots(t_{N-1}-t_N)\,t_N \quad .$$

The inverse is

$$M^{-1} = \frac{1}{2D} \begin{pmatrix} \frac{1}{t_1-t_2} & -\frac{1}{t_1-t_2} & 0 & \cdots & & & & \\ -\frac{1}{t_1-t_2} & \frac{t_1-t_3}{(t_1-t_2)(t_2-t_3)} & -\frac{1}{t_2-t_3} & 0 & \cdots & & & \\ \vdots & \vdots & \ddots & & & & & \\ \cdots & 0 & -\frac{1}{t_{n-1}-t_n} & \frac{t_{n-1}-t_{n+1}}{(t_{n-1}-t_n)(t_n-t_{n+1})} & -\frac{1}{t_n-t_{n+1}} & 0 & \cdots & \\ & & & \ddots & & & & \\ & & & & \cdots & 0 & -\frac{1}{t_{N-1}-t_N} & \frac{t_{N-1}}{t_N}\,\frac{1}{t_{N-1}-t_N} \end{pmatrix} \quad .$$

This yields the general result

$$\sum_{j,j'=1}^{N} M^{-1}_{j,j'}(t_1,\ldots,t_N)\, x_j\, x_{j'} = \sum_{j=1}^{N} \left(\frac{1}{t_{j-1}-t_j} + \frac{1}{t_j - t_{j+1}} \right) x_j^2 - \frac{2}{t_j - t_{j+1}}\, x_j\, x_{j+1} \quad ,$$

where $t_0 \equiv \infty$ and $t_{N+1} \equiv 0$. Now consider the conditional probability density

$$\begin{aligned} P(x_1,t_1\,|\,x_2,t_2\,;\,\ldots\,;\,x_N,t_N) &= \frac{P(x_1,t_1\,;\,\ldots\,;\,x_N,t_N)}{P(x_2,t_2\,;\,\ldots\,;\,x_N,t_N)} \\ &= \frac{\mathsf{det}^{1/2}\, 2\pi M(t_2,\ldots,t_N)}{\mathsf{det}^{1/2}\, 2\pi M(t_1,\ldots,t_N)}\, \frac{\exp\Big\{ -\frac{1}{2}\sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1,\ldots,t_N)\, x_j\, x_{j'} \Big\}}{\exp\Big\{ -\frac{1}{2}\sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2,\ldots,t_N)\, x_k\, x_{k'} \Big\}} \end{aligned}$$

We have

$$\begin{aligned} \sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1,\ldots,t_N)\, x_j\, x_{j'} &= \left(\frac{1}{t_0-t_1} + \frac{1}{t_1-t_2} \right) x_1^2 - \frac{2}{t_1-t_2}\, x_1\, x_2 + \left(\frac{1}{t_1-t_2} + \frac{1}{t_2-t_3} \right) x_2^2 + \ldots \\ \sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2,\ldots,t_N)\, x_k\, x_{k'} &= \left(\frac{1}{t_0-t_2} + \frac{1}{t_2-t_3} \right) x_2^2 + \ldots \end{aligned}$$

Subtracting, and evaluating the ratio to get the conditional probability density, we find

$$P(x_1,t_1\,|\,x_2,t_2\,;\,\ldots\,;\,x_N,t_N) = \frac{1}{\sqrt{4\pi D(t_1-t_2)}}\, e^{-(x_1-x_2)^2/4D(t_1-t_2)} \quad ,$$

which depends only on $\{x_1,t_1,x_2,t_2\}$, *i.e.* on the current and most recent data, and not on any data before the time t_2. Note the normalization:

$$\int\limits_{-\infty}^{\infty} dx_1\, P(x_1,t_1\,|\,x_2,t_2\,;\,\ldots\,;\,x_N,t_N) = 1 \quad .$$

(9.3) Consider a discrete one-dimensional random walk where the probability to take a step of length 1 in either direction is $\frac{1}{2}p$ and the probability to take a step of length 2 in either direction is $\frac{1}{2}(1-p)$. Define the generating function

$$\hat{P}(k,t) = \sum_{n=-\infty}^{\infty} P_n(t)\, e^{-ikn} \quad ,$$

where $P_n(t)$ is the probability to be at position n at time t. Solve for $\hat{P}(k,t)$ and provide an expression for $P_n(t)$. Evaluate $\sum_n n^2\, P_n(t)$.

Solution:

We have the master equation

$$\frac{dP_n}{dt} = \tfrac{1}{2}(1-p)\, P_{n+2} + \tfrac{1}{2}p\, P_{n+1} + \tfrac{1}{2}p\, P_{n-1} + \tfrac{1}{2}(1-p)\, P_{n-2} - P_n \quad .$$

Upon Fourier transforming,

$$\frac{d\hat{P}(k,t)}{dt} = \Big[(1-p)\cos(2k) + p\cos(k) - 1\Big]\,\hat{P}(k,t) \quad ,$$

with the solution

$$\hat{P}(k,t) = e^{-\lambda(k)\,t}\,\hat{P}(k,0) \quad ,$$

where

$$\lambda(k) = 1 - p\cos(k) - (1-p)\cos(2k) \quad .$$

One then has

$$P_n(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{ikn}\,\hat{P}(k,t) \quad .$$

The average of n^2 is given by

$$\big\langle n^2 \big\rangle_t = -\frac{\partial^2 \hat{P}(k,t)}{\partial k^2}\bigg|_{k=0} = \Big[\lambda''(0)\,t - \lambda'(0)^2\,t^2\Big] = \big(4-3p\big)\,t \quad .$$

Note that $\hat{P}(0,t) = 1$ for all t by normalization.

(9.4) Numerically simulate the one-dimensional Wiener and Cauchy processes discussed in ch. 9 of the lecture notes, and produce a figure similar.

Solution:

Most computing languages come with a random number generating function which produces uniform deviates on the interval $x \in [0,1]$. Suppose we have a prescribed function $y(x)$. If x is distributed uniformly on $[0,1]$, how is y distributed? Clearly

$$\big|p(y)\,dy\big| = \big|p(x)\,dx\big| \qquad \Rightarrow \qquad p(y) = \left|\frac{dx}{dy}\right| p(x) \quad ,$$

where for the uniform distribution on the unit interval we have $p(x) = \Theta(x)\,\Theta(1-x)$. For example, if $y = -\ln x$, then $y \in [0,\infty]$ and $p(y) = e^{-y}$ which is to say y is exponentially distributed. Now suppose we want to specify $p(y)$. We have

$$\frac{dx}{dy} = p(y) \qquad \Rightarrow \qquad x = F(y) = \int\limits_{y_0}^{y} \! d\tilde{y}\, p(\tilde{y}) \quad ,$$

where y_0 is the minimum value that y takes. Therefore, $y = F^{-1}(x)$, where F^{-1} is the inverse function.

To generate normal (Gaussian) deviates with a distribution $p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon)$, we have

$$F(y) = \frac{1}{\sqrt{4\pi D\varepsilon}} \int\limits_{-\infty}^{y} \! d\tilde{y}\, e^{-\tilde{y}^2/4D\varepsilon} = \tfrac{1}{2} + \tfrac{1}{2}\,\text{erf}\bigg(\frac{y}{\sqrt{4D\varepsilon}}\bigg) \quad .$$

We now have to invert the error function, which is slightly unpleasant.

A slicker approach is to use the *Box-Muller* method, which used a two-dimensional version of the above transformation,

$$p(y_1,y_2) = p(x_1,x_2) \left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right| \quad .$$

This has an obvious generalization to higher dimensions. The transformation factor is the Jacobian determinant. Now let x_1 and x_2 each be uniformly distributed on $[0,1]$, and let

$$x_1 = \exp\bigg(-\frac{y_1^2+y_2^2}{4D\varepsilon}\bigg) \qquad\qquad y_1 = \sqrt{-4D\varepsilon \ln x_1}\,\cos(2\pi x_2)$$

$$x_2 = \frac{1}{2\pi} \tan^{-1}(y_2/y_1) \qquad\qquad y_2 = \sqrt{-4D\varepsilon \ln x_1}\,\sin(2\pi x_2)$$

Then

$$\frac{\partial x_1}{\partial y_1} = -\frac{y_1\,x_1}{2D\varepsilon} \qquad\qquad \frac{\partial x_2}{\partial y_1} = -\frac{1}{2\pi}\,\frac{y_2}{y_1^2+y_2^2}$$

$$\frac{\partial x_1}{\partial y_2} = -\frac{y_2\,x_1}{2D\varepsilon} \qquad\qquad \frac{\partial x_2}{\partial y_2} = \frac{1}{2\pi}\,\frac{y_1}{y_1^2+y_2^2}$$

and therefore the Jacobian determinant is

$$J = \left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right| = \frac{1}{4\pi D\varepsilon}\, e^{-(y_1^2+y_2^2)/4D\varepsilon} = \frac{e^{-y_1^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \cdot \frac{e^{-y_2^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \quad ,$$

which says that y_1 and y_2 are each independently distributed according to the normal distribution, which is $p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon)$. Nifty!

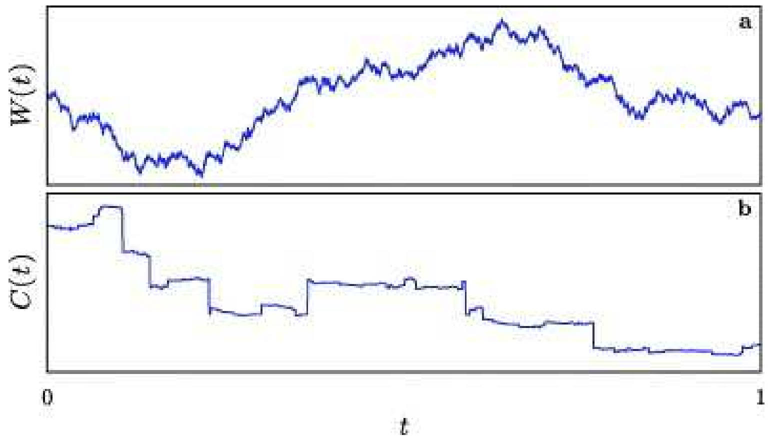

Figure 1: (a) Wiener process sample path $W(t)$. (b) Cauchy process sample path $C(t)$. From K. Jacobs and D. A. Steck, *New J. Phys.* **13**, 013016 (2011).

For the Cauchy distribution, with

$$p(y) = \frac{1}{\pi}\,\frac{\varepsilon}{y^2+\varepsilon^2} \quad ,$$

we have

$$F(y) = \frac{1}{\pi}\int_{-\infty}^{y} d\tilde{y}\,\frac{\varepsilon}{\tilde{y}^2+\varepsilon^2} = \tfrac{1}{2} + \tfrac{1}{\pi}\tan^{-1}(y/\varepsilon) \quad ,$$

and therefore

$$y = F^{-1}(x) = \varepsilon\tan\left(\pi x - \tfrac{\pi}{2}\right) \quad .$$

(9.5) A *Markov chain* is a probabilistic process which describes the transitions of discrete stochastic variables in time. Let $P_i(t)$ be the probability that the system is in state i at time t. The time evolution equation for the probabilities is

$$P_i(t+1) = \sum_j Y_{ij}\, P_j(t) \quad .$$

Thus, we can think of $Y_{ij} = P(i\,,\,t+1\,|\,j\,,\,t)$ as the *conditional probability* that the system is in state i at time $t+1$ given that it was in state j at time t. Y is called the *transition matrix*. It must satisfy $\sum_i Y_{ij} = 1$ so that the total probability $\sum_i P_i(t)$ is conserved.

Suppose I have two bags of coins. Initially bag A contains two quarters and bag B contains five dimes. Now I do an experiment. Every minute I exchange a random coin chosen from each of the bags. Thus the number of coins in each bag does not fluctuate, but their values do fluctuate.

(a) Label all possible states of this system, consistent with the initial conditions. (*I.e.* there are always two quarters and five dimes shared among the two bags.)

(b) Construct the transition matrix Y_{ij}.

(c) Show that the total probability is conserved is $\sum_i Y_{ij} = 1$, and verify this is the case for your transition matrix Y. This establishes that $(1, 1, \ldots, 1)$ is a left eigenvector of Y corresponding to eigenvalue $\lambda = 1$.

(d) Find the eigenvalues of Y.

(e) Show that as $t \to \infty$, the probability $P_i(t)$ converges to an equilibrium distribution $P_i^{\rm eq}$ which is given by the right eigenvector of i corresponding to eigenvalue $\lambda = 1$. Find $P_i^{\rm eq}$, and find the long time averages for the value of the coins in each of the bags.

Solution :

(a) There are three possible states consistent with the initial conditions. In state $|\,1\,\rangle$, bag A contains two quarters and bag B contains five dimes. In state $|\,2\,\rangle$, bag A contains a quarter and a dime while bag B contains a quarter and five dimes. In state $|\,3\,\rangle$, bag A contains two dimes while bag B contains three dimes and two quarters. We list these states in the table below, along with their degeneracies. The degeneracy of a state is the number of configurations consistent with the state label. Thus, in state $|\,2\,\rangle$ the first coin in bag A could be a quarter and the second a dime, or the first could be a dime and the second a quarter. For bag B, any of the five coins could be the quarter.

(b) To construct Y_{ij}, note that transitions out of state $|\,1\,\rangle$, *i.e.* the elements Y_{i1}, are particularly simple. With probability 1, state $|\,1\,\rangle$ always evolves to state $|\,2\,\rangle$. Thus, $Y_{21} = 1$ and $Y_{11} = Y_{31} = 0$. Now consider transitions out of state $|\,2\,\rangle$. To get to state $|\,1\,\rangle$, we need to choose the D from bag A (probability $\frac{1}{2}$) and the Q from bag B (probability $\frac{1}{5}$). Thus, $Y_{12} = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$. For transitions back to state $|\,2\,\rangle$, we could choose the Q from bag A (probability $\frac{1}{2}$) if we also chose the Q from bag B (probability $\frac{1}{5}$). Or we could choose the D from bag A (probability $\frac{1}{2}$) and one of the D's from bag B (probability $\frac{4}{5}$). Thus, $Y_{22} = \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{1}{2}$. Reasoning thusly, one obtains the transition matrix,

$$Y = \begin{pmatrix} 0 & \frac{1}{10} & 0 \\ 1 & \frac{1}{2} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix} \quad .$$

Note that $\sum_i Y_{ij} = 1$.

(c) Our explicit form for Y confirms the sum rule $\sum_i Y_{ij} = 1$ for all j. Thus, $\vec{L}^1 = (1\;1\;1)$ is a left eigenvector of Y with eigenvalue $\lambda = 1$.

$\lvert j \rangle$	bag A	bag B	$g_j^{\rm A}$	$g_j^{\rm B}$	$g_j^{\rm TOT}$
$\lvert 1 \rangle$	QQ	DDDDD	1	1	1
$\lvert 2 \rangle$	QD	DDDDQ	2	5	10
$\lvert 3 \rangle$	DD	DDDQQ	1	10	10

Table 1: States and their degeneracies.

(d) To find the other eigenvalues, we compute the characteristic polynomial of Y and find, easily,

$$P(\lambda) = \det(\lambda\,\mathbb{I} - Y) = \lambda^3 - \tfrac{11}{10}\,\lambda^2 + \tfrac{1}{25}\,\lambda + \tfrac{3}{50} \quad .$$

This is a cubic, however we already know a root, *i.e.* $\lambda = 1$, and we can explicitly verify $P(\lambda = 1) = 0$. Thus, we can divide $P(\lambda)$ by the monomial $\lambda - 1$ to get a quadratic function, which we can factor. One finds after a small bit of work,

$$\frac{P(\lambda)}{\lambda - 1} = \lambda^2 - \tfrac{3}{10}\,\lambda - \tfrac{3}{50} = \big(\lambda - \tfrac{3}{10}\big)\big(\lambda + \tfrac{1}{5}\big) \quad .$$

Thus, the eigenspectrum of Y is $\lambda_1 = 1$, $\lambda_2 = \frac{3}{10}$, and $\lambda_3 = -\frac{1}{5}$.

(e) We can decompose Y into its eigenvalues and eigenvectors, like we did in problem (1). Write

$$Y_{ij} = \sum_{\alpha=1}^{3} \lambda_\alpha R_i^\alpha \, L_j^\alpha \quad .$$

Now let us start with initial conditions $P_i(0)$ for the three configurations. We can always decompose this vector in the right eigenbasis for Y, *viz.*

$$P_i(t) = \sum_{\alpha=1}^{3} C_\alpha(t)\, R_i^\alpha \quad ,$$

The initial conditions are $C_\alpha(0) = \sum_i L_i^\alpha \, P_i(0)$. But now using our eigendecomposition of Y, we find that the equations for the discrete time evolution for each of the C_α decouple:

$$C_\alpha(t+1) = \lambda_\alpha C_\alpha(t) \quad .$$

Clearly as $t \to \infty$, the contributions from $\alpha = 2$ and $\alpha = 3$ get smaller and smaller, since $C_\alpha(t) = \lambda_\alpha^t \, C_\alpha(0)$, and both λ_2 and λ_3 are smaller than unity in magnitude. Thus, as $t \to \infty$ we have $C_1(t) \to C_1(0)$, and $C_{2,3}(t) \to 0$. Note $C_1(0) = \sum_i L_i^1 \, P_i(0) = \sum_i P_i(0) = 1$, since $\vec{L}^1 = (1\;1\;1)$. Thus, we obtain $P_i(t \to \infty) \to R_i^1$, the components of the eigenvector $\vec{R}^1$. It is not too hard to explicitly compute the eigenvectors:

$$\vec{L}^1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad \vec{L}^2 = \begin{pmatrix} 10 & 3 & -4 \end{pmatrix} \qquad \vec{L}^3 = \begin{pmatrix} 10 & -2 & 1 \end{pmatrix}$$

$$\vec{R}^1 = \tfrac{1}{21}\begin{pmatrix} 1 \\ 10 \\ 10 \end{pmatrix} \qquad \vec{R}^2 = \tfrac{1}{35}\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \qquad \vec{R}^3 = \tfrac{1}{15}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad .$$

Thus, the equilibrium distribution $P_i^{\rm eq} = \lim_{t\to\infty} P_i(t)$ satisfies detailed balance:

$$P_j^{\rm eq} = \frac{g_j^{\rm TOT}}{\sum_l g_l^{\rm TOT}} \quad .$$

Working out the average coin value in bags A and B under equilibrium conditions, one finds $A = \frac{200}{7}$ and $B = \frac{500}{7}$ (cents), and B/A is simply the ratio of the number of coins in bag B to the number in bag A. Note $A + B = 100$ cents, as the total coin value is conserved.