PHYSICS 210A : EQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #7

(1) The Hamiltonian for the three state ($\mathbb{Z}_3$) clock model is written

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j \ ,$$

where each local unit vector $\hat{\boldsymbol{n}}_i$ can take one of three possible values:

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}} \quad , \quad \hat{\boldsymbol{n}} = -\tfrac{1}{2}\hat{\boldsymbol{x}} + \tfrac{\sqrt{3}}{2}\hat{\boldsymbol{y}} \quad , \quad \hat{\boldsymbol{n}} = -\tfrac{1}{2}\hat{\boldsymbol{x}} - \tfrac{\sqrt{3}}{2}\hat{\boldsymbol{y}} \ .$$

(a) Consider the $\mathbb{Z}_3$ clock model on a lattice of coordination number z. Make the mean field assumption $\langle \hat{\boldsymbol{n}}_i \rangle = m\hat{\boldsymbol{x}}$. Expanding the Hamiltonian to linear order in the fluctuations, derive the mean field Hamiltonian for this model $\hat{H}_{\text{MF}}$.

(b) Rescaling $\theta = k_{\text{B}}T/zJ$ and $f = F/NzJ$, where N is the number of sites, find $f(m,\theta)$.

(c) Find the mean field equation.

(d) Is the transition second order or first order?

(e) Show that this model is equivalent to the three state Potts model. Is the $\mathbb{Z}_4$ clock model equivalent to the four state Potts model? Why or why not?

(2) Consider a four-state Ising model on a cubic lattice with Hamiltonian

$$\hat{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i \quad ,$$

where each spin variable S_i takes on one of four possible values: $S_i \in \{-2, -1, +1, +2\}$, and the first sum is over all nearest-neighbor pairs of the lattice (*i.e.* over all unique links). Note there is no $S_i = 0$ state.

(a) What is the mean field Hamiltonian $\hat{H}_{\text{MF}}$?

(b) Find the mean field free energy per site $f(\theta, h, m)$, where $m = \langle S_i \rangle$, $\theta = k_{\text{B}}T/zJ$, $h = H/zJ$, and $f = F/NzJ$. Here z is the lattice coordination number.

(c) Find the mean field equation relating m, θ, and h.

(d) Expand f to fourth order in m, retaining terms only to first order in h, and working to lowest order in $\theta - \theta_{\text{c}}$. What is θ_{c}?

(e) If $J/k_{\text{B}} = 100\,\text{K}$, what is the critical temperature T_{c}?

(3) Consider the free energy

$$f(\theta, m) = f_0 + \tfrac{1}{2}am^2 + \tfrac{1}{4}bm^4 + \tfrac{1}{8}dm^8$$

with $d > 0$. Note there is an octic term but no sextic term. Derive results corresponding to those in fig. 7.16 of the lecture notes. Find the equation of the first order line in the $(a/d, b/d)$ plane. Also identify the region in parameter space where there exist metastable local minima in the free energy (curve E in fig. 7.16).

(4) Consider the $\mathsf{U}(1)$ Ginsburg-Landau theory with

$$F = \int d^d\!\boldsymbol{r}\, \Big[\tfrac{1}{2} a\, |\Psi|^2 + \tfrac{1}{4} b\, |\Psi|^4 + \tfrac{1}{2}\kappa\, |\boldsymbol{\nabla}\Psi|^2 \Big] \,.$$

Here $\Psi(\boldsymbol{r})$ is a complex-valued field, and both b and κ are positive. This theory is appropriate for describing the transition to superfluidity. The order parameter is $\langle \Psi(\boldsymbol{r})\rangle$. Note that the free energy is a functional of the two independent fields $\Psi(\boldsymbol{r})$ and $\Psi^*(\boldsymbol{r})$, where Ψ^* is the complex conjugate of Ψ. Alternatively, one can consider F a functional of the real and imaginary parts of Ψ.

(a) Show that one can rescale the field Ψ and the coordinates $\boldsymbol{r}$ so that the free energy can be written in the form

$$F = \varepsilon_0 \int d^d\!x\, \Big[\pm \tfrac{1}{2}\, |\psi|^2 + \tfrac{1}{4}\, |\psi|^4 + \tfrac{1}{2}\, |\boldsymbol{\nabla}\psi|^2 \Big] \,,$$

where ψ and $\boldsymbol{x}$ are dimensionless, ε_0 has dimensions of energy, and where the sign on the first term on the RHS is $\mathsf{sgn}(a)$. Find ε_0 and the relations between Ψ and ψ and between $\boldsymbol{r}$ and $\boldsymbol{x}$.

(b) By extremizing the functional $F[\psi, \psi^*]$ with respect to ψ^*, find a partial differential equation describing the behavior of the order parameter field $\psi(\boldsymbol{x})$.

(c) Consider a two-dimensional system ($d = 2$) and let $a < 0$ (*i.e.* $T < T_c$). Consider the case where $\psi(\boldsymbol{x})$ describe a *vortex* configuration: $\psi(\boldsymbol{x}) = f(r)\, e^{i\phi}$, where (r, ϕ) are two-dimensional polar coordinates. Find the ordinary differential equation for $f(r)$ which extremizes F.

(d) Show that the free energy, up to a constant, may be written as

$$F = 2\pi\varepsilon_0 \int\limits_0^R dr\, r\, \left[\tfrac{1}{2} \big(f'\big)^2 + \frac{f^2}{2r^2} + \tfrac{1}{4}\big(1 - f^2\big)^2 \right] \,,$$

where R is the radius of the system, which we presume is confined to a disk. Consider a *trial solution* for $f(r)$ of the form

$$f(r) = \frac{r}{\sqrt{r^2 + a^2}} \,,$$

where a is the variational parameter. Compute $F(a, R)$ in the limit $R \to \infty$ and extremize with respect to a to find the optimum value of a within this variational class of functions.