PHYSICS 210A : EQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #2

(1) Consider a q-state generalization of the Kac ring model in which $\mathbb{Z}_q$ spins rotate around an N-site ring which contains a fraction $x = N_{\rm F}/N$ of flippers on its links. Each flipper cyclically rotates the spin values: $1 \to 2 \to 3 \to \cdots \to q \to 1$ (hence the clock model symmetry $\mathbb{Z}_q$).

(a) What is the Poincare recurrence time?

(b) Make the *Stosszahlansatz*, *i.e.* assume the spin flips are stochastic random processes. Then one has

$$P_\sigma(t+1) = (1-x)\, P_\sigma(t) + x\, P_{\sigma-1}(t) \ ,$$

where $P_0 \equiv P_q$. This defines a Markov chain

$$P_\sigma(t+1) = Q_{\sigma\sigma'}\, P_{\sigma'}(t) \ .$$

Decompose the transition matrix Q into its eigenvectors. *Hint:* The matrix may be diagonalized by a simple Fourier transform.

(c) The eigenvalues of Q may be written as $\lambda_\alpha = e^{-1/\tau_\alpha}\, e^{-i\delta_\alpha}$, where τ_α is a relaxation time and δ_α is a phase. Find the spectrum of relaxation times. What is the longest finite relaxation time?

(d) Suppose all the spins are initially in the state $\sigma = q$. Write down an expression for $P_\sigma(t)$ for all subsequent times $t \in \mathbb{Z}^+$. Plot your results for different values of x and q.

(2) Consider a system with K possible states $|\, i\,\rangle$, with $i \in \{1, \ldots, K\}$, where the transition rate W_{ij} between any two states is the same, with $W_{ij} = \gamma > 0$.

(a) Find the matrix $\varGamma_{ij}$ governing the master equation $\dot{P}_i = -\varGamma_{ij}\, P_j$.

(b) Find all the eigenvalues and eigenvectors of $\varGamma$. What is the equilibrium distribution?

(c) Now suppose there are $2K$ possible states $|\, i\,\rangle$, with $i \in \{1, \ldots, 2K\}$, and the transition rate matrix is

$$W_{ij} = \begin{cases} \alpha & \text{if} \quad (-1)^{ij} = +1 \\ \beta & \text{if} \quad (-1)^{ij} = -1 \ , \end{cases}$$

with $\alpha, \beta > 0$. Repeat parts (a) and (b) for this system.

(3) A generalized two-dimensional cat map can be defined by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & p \\ q & pq+1 \end{pmatrix}}^{M} \begin{pmatrix} x \\ y \end{pmatrix} \ \textsf{mod}\ \mathbb{Z}^2 \ ,$$

where p and q are integers. Here $x, y \in [0, 1]$ are two real numbers on the unit interval, so $(x, y) \in \mathbb{T}^2$ lives on a two-dimensional torus. The inverse map is

$$M^{-1} = \begin{pmatrix} pq+1 & -p \\ -q & q \end{pmatrix} .$$

Note that $\det M = 1$.

(a) Consider the action of this map on a pixelated image of size $(lK)\times(lK)$, where $l \sim 4-10$ and $K \sim 20 - 100$. Starting with an initial state in which all the pixels in the left half of the array are "on" and the others are all "off", iterate the image with the generalized cat map, and compute at each state the entropy $S = -\sum_{\boldsymbol{r}} p_{\boldsymbol{r}} \ln p_{\boldsymbol{r}}$, where the sum is over the K^2 different $l \times l$ subblocks, and $p_{\boldsymbol{r}}$ is the probability to find an "on" pixel in subblock $\boldsymbol{r}$. (Take $p = q = 1$ for convenience, though you might want to explore other values.)

Now consider a three-dimensional generalization (Chen *et al.*, *Chaos, Solitons, and Fractals* **21**, 749 (2004)), with

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ \mathsf{mod}\ \mathbb{Z}^3 ,$$

which is a discrete automorphism of $\mathbb{T}^3$, the three-dimensional torus. Again, we require that both M and M^{-1} have integer coefficients. This can be guaranteed by writing

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p_x \\ 0 & q_x & p_x q_x + 1 \end{pmatrix} \quad , \quad M_y = \begin{pmatrix} 1 & 0 & p_y \\ 0 & 1 & 0 \\ q_y & 0 & p_y q_y + 1 \end{pmatrix} \quad , \quad M_z = \begin{pmatrix} 1 & p_z & 0 \\ q_z & p_z q_z + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and taking $M = M_x M_y M_z$, reminiscent of how we build a general $\mathsf{O}(3)$ rotation from a product of three $\mathsf{O}(2)$ rotations about different axes.

(b) Find M and M^{-1} when $p_x = q_x = p_y = q_y = p_z = q_z = 1$.

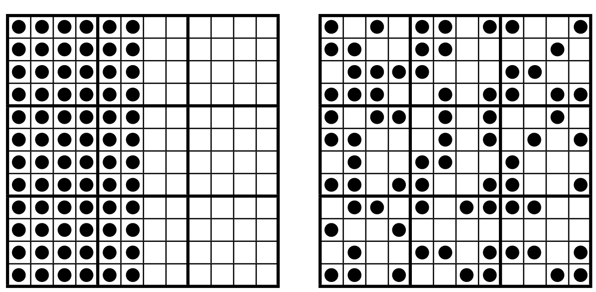

Figure 1: Two-dimensional cat map on a 12×12 square array with $l = 4$ and $K = 3$ shown. Left: initial conditions at $t = 0$. Right: possible conditions at some later time $t > 0$. Within each $l \times l$ cell $\boldsymbol{r}$, the occupation probability $p_{\boldsymbol{r}}$ is computed. The entropy $-p_{\boldsymbol{r}} \log_2 p_{\boldsymbol{r}}$ is then averaged over the K^2 cells.

(c) Repeat part (a) for this three-dimensional generalized cat map, computing the entropy by summing over the K^3 different $l \times l \times l$ subblocks.

(d) 100 quatloos extra credit if you find a way to show how a three dimensional object (a ball, say) evolves under this map. Is it Poincaré recurrent?

(4) Consider a d-dimensional ideal gas with dispersion $\varepsilon(\boldsymbol{p}) = A|\boldsymbol{p}|^{\alpha}$, with $\alpha > 0$. Find the density of states $D(E, V, N)$, the statistical entropy $S(E, V, N)$, the equation of state $p = p(T, V, N)$, the heat capacity at constant volume $C_{V,N}(T, V, N)$, and the heat capacity at constant pressure $C_{p,N}(T, V, N)$. (Particle number N is held constant.) Recall the thermodynamic relation

$$dS = \frac{1}{T}\,dE + \frac{p}{T}\,dV - \frac{\mu}{T}\,dN \quad ,$$

(5) Write a well-defined expression for the greatest possible number expressible using only five symbols. *Examples:* $1+2+3$, 10^{100}, $\Gamma(99)$. `[50 quatloos extra credit]`