PHYSICS 210A : STATISTICAL PHYSICS
FINAL EXAMINATION

(1) Provide clear, accurate, and brief answers for each of the following:

(a) For the free energy density $f = \frac{1}{2}am^2 - \frac{1}{3}ym^3 + \frac{1}{4}bm^4$, what does it mean to say that 'a first order transition preempts the second order transition'?
`[5 points]`

(b) A system of noninteracting bosons has a power law dispersion $\varepsilon(\boldsymbol{k}) = A\,k^\sigma$. What is the condition on the power σ and the dimension d of space such that Bose condensation will occur at some finite temperature?
`[5 points]`

(c) Sketch what the pair distribution function $g(r)$ should look like for a fluid composed of infinitely hard spheres of diameter a. How does $g(r)$ change with temperature?
`[5 points]`

(d) For the cluster γ shown in Fig. 1, identify the symmetry factor s_γ, the lowest order virial coefficient B_j to which this contributes, and write an expression for the cluster integral $b_\gamma(T)$ in terms of the Mayer function.
`[5 points]`

(e) Explain the following terms in the context of dynamical systems: *recurrent*, *ergodic*, and *mixing*. How are these classifications arranged hierarchically?
`[5 points]`

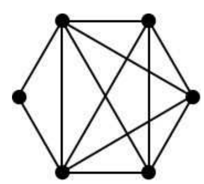

Figure 1: The connected cluster γ for problem (1d).

(2) Consider a gas of spinless bosons in $d = 3$ dimensions with dispersion $\varepsilon(\boldsymbol{p}) = \boldsymbol{p}^2/2m$ and bulk number density n. The gas is in equilibrium with a $d = 2$ surface, and the surface bosons have dispersion $\varepsilon_{2\text{d}}(p_x, p_y) = (p_x^2 + p_y^2)/2m + \Delta$ with $\Delta > 0$. Note that $\text{Li}_1(z) = -\log(1-z)$.

(a) For $T > T_\text{c}$, where T_c is the bulk critical temperature for Bose-Einstein condensation, write down relations between (i) the bulk density n, temperature T, and fugacity z, and (ii) the surface density $n_{2\text{d}}$, T, and z. Your expressions may also involve m, Δ, and numeral and physical constants.
`[5 points]`

(b) For $T < T_\text{c}$, write down relations between (i) the bulk density n, temperature T, and bulk condensate density n_0, and (ii) the surface density $n_{2\text{d}}$, T, and n_0. Your expressions

may also involve m, Δ, and numeral and physical constants.
[10 points]

(c) What is $T_{\rm c}$? Find an equation relating $n_{\rm 2d}(T_{\rm c})$ to n, Δ, m, and physical constants.
[5 points]

(d) What is the $T \to 0$ limit of the surface density $n_{\rm 2d}$? Interpret your result. Do the surface bosons ever condense? Why or why not? What do you think happens if $\Delta < 0$?
[5 points]

Important: You may regard the bulk number density to be fixed at n regardless of the surface number density $n_{\rm 2d}$ because in the thermodynamic limit the bulk contains vastly more particles than the surface.

(3) The Hamiltonian for the one-dimensional p-state clock model is

$$\hat{H} = -J \sum_i \cos\left(\frac{2\pi(n_i - n_{i+1})}{p}\right) = \sum_i E_{n_i, n_{i+1}} \quad ,$$

where on each site one has $n_i \in \{0, 1, \ldots, p-1\}$. Here you are invited to consider the case $p = 4$. The interaction energy between neighboring clock spins with values n and n' is then

$$E_{n,n'} = \begin{pmatrix} -J & 0 & J & 0 \\ 0 & -J & 0 & J \\ J & 0 & -J & 0 \\ 0 & J & 0 & -J \end{pmatrix} \quad .$$

(a) Write down the transfer matrix $T_{n,n'}$ at temperature T.
[9 points]

(b) Find the eigenvalues of T. As a helpful hint, note that the (normalized) eigenvectors of the matrix

$$M = \begin{pmatrix} a & 1 & b & 1 \\ 1 & a & 1 & b \\ b & 1 & a & 1 \\ 1 & b & 1 & a \end{pmatrix}$$

are

$$\boldsymbol{\psi}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_4 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad .$$

[8 points]

(c) Find the free energy per site in the thermodynamic limit.
[8 points]

(d) Let $\phi_i = \delta_{n_i,1} - \frac{1}{4}$. This serves as an order parameter since $\langle \phi_i \rangle = 0$ in the disordered phase where each n value is equally likely and hence $\langle \delta_{n_i,1} \rangle = \frac{1}{4}$. Using the transfer matrix formalism, find the correlator $C(m) = \langle \phi_1 \, \phi_{1+m} \rangle$. What is the correlation length $\xi(T)$?
`[100 quatloos extra credit]`

(4) Now consider the mean field phase transition of the $p = 4$ clock model, with

$$\hat{H} = -J \sum_{\langle ij \rangle} \cos\!\left(\frac{2\pi(n_i - n_j)}{4}\right) - H \sum_i \left(\delta_{n_i,0} - \tfrac{1}{4}\right) \quad ,$$

with $n_i \in \{0,1,2,3\}$ on each site i. Find the variational free energy from the normalized single site density matrix

$$\begin{aligned}
\varrho(n) &= \frac{e^{u\cos(2\pi n/4)}}{2 + 2\cosh u} \\
&= \frac{e^{u}}{2 + 2\cosh u}\,\delta_{n,0} + \frac{1}{2 + 2\cosh u}\,\delta_{n,1} + \frac{e^{-u}}{2 + 2\cosh u}\,\delta_{n,2} + \frac{1}{2 + 2\cosh u}\,\delta_{n,3} \quad .
\end{aligned}$$

(a) Find $E = \mathsf{Tr}\,(\hat{H}\,\varrho_N^{\rm var})$, where $\varrho_N = \prod_{i=1}^N \varrho(n_i)$. You should assume a Bravais lattice with of coordination number z with nearest neighbor interactions only.
`[6 points]`

(b) Find $S = -k_{\rm B}\,\mathsf{Tr}\left(\varrho_N^{\rm var} \log \varrho_N^{\rm var}\right)$.
`[6 points]`

(c) Find the dimensionless free energy per site $f(u,\theta,h) = F/NzJ$, with $\theta = k_{\rm B}T/zJ$ and $h = H/zJ$ the dimensionless temperature and symmetry-breaking external field.
`[6 points]`

(d) Find $\theta_{\rm c}$ and $\phi(h)$ above $\theta_{\rm c}$, where $\phi = \langle \delta_{n_i,1} - \frac{1}{4} \rangle$ is the order parameter.
`[7 points]`