Physics 152B/232 Spring 2025

Problem Set 5

Problem 1

Consider a generalized Hubbard model described by the Hamiltonian

$$H=-t\sum_{\substack{<ij>\\ \sigma}}(c^+_{i\sigma}c_{j\sigma}+h.c.)+U\sum_i n_{i\uparrow}n_{i\downarrow}+J\sum_{\substack{<ij>\\ \sigma\sigma'}}c^+_{i\sigma}c^+_{j\sigma'}c_{i\sigma'}c_{j\sigma}$$

Assume U>0 and J>0 and there are only 2 sites in the system.

(a) Find the energies of all the states with two electrons in the system.

(b) Find the condition that J has to satisfy for the lowest two—electron state to be ferromagnetic for given U and t.

(c) Find a simplified form for that condition valid in the limit of large U/t.

(d) For extra credit: include an interaction $V\sum_{<ij>}n_i n_j$, with $n_i = n_{i\uparrow}+n_{i\downarrow}$ repeat the above.

Problem 2

Consider a superconducting cylinder of radius R=4cm and height h=20cm and carrier density 4×10^{22} electrons/cm^3 in the absence of applied magnetic field, initially at rest. Then, it is put into rotation around its symmetry axis with angular frequency ω.

(a) Show that a magnetic field B develops in its interior, find an expression for it.
Hint: follow the derivation given in lecture.

(b) Find an expression for the current density near the surface that gives rise to this magnetic field, assuming that the current circulates only in a layer of thickness λ_L next to the surface, with λ_L <<R. Assume h>>R.
Hint: use Ampere's law.

(c) What is the magnitude of the magnetic field in Gauss for ω=20,000 s^{-1} ?

(d) Find approximately the total current I (in Amps) that circulates when the angular velocity ω is 20,000 s^{-1} .

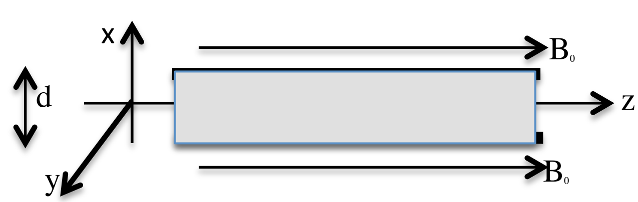
Problem 3

Consider an infinite slab of superconducting material of thickness d in an applied magnetic field B_0 parallel to its surfaces. The slab is parallel to the zy plane, its center is at x=0, the applied magnetic field is along the z direction. Assume the London penetration depth is $\lambda_L = 300 A$

(a) By solving the London equation, find an expression for the magnetic field inside the slab as function of position.

(b) What is the minimum thickness d (in cm) so that the magnetic field at the center of the slab is smaller than $B_0/200$?

(c) For B_0=500 Gauss (0.05T) find the speed of electrons v (in cm/s) at the surface and at the center of the slab for the thickness found in (b). Make a qualitative plot of v versus x. In which direction does the current flow?



Problem 4 [for 232 students; extra credit for 152B students]

Consider the reduced Hamiltonian

$$H = \sum_{k\sigma} (\varepsilon_k - \mu) c^+_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} c^+_{k\uparrow} c^+_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

with the pairing interaction derived in HW4, problem 2:

$$V_{kk'} = \frac{U}{N} - 2\frac{\alpha}{N}(\varepsilon_k + \varepsilon_{k'})$$

Using the BCS gap equation derived in class

$$\Delta_k = -\sum_{k'} V_{kk'} \Delta_{k'} \frac{1-2f(E_{k'})}{2E_{k'}}$$

(a) Show that the energy gap can be written as

$$\Delta_k = a + b\ \varepsilon_k$$

and give equations for a and b.

(b) Defining the quantities

$$I_n = \frac{1}{N}\sum_k (\varepsilon_k)^n \frac{1-2f(|\varepsilon_k - \mu|)}{|\varepsilon_k - \mu|}$$

find an equation that determines the critical temperature.