PHYSICS 140A : STATISTICAL PHYSICS
HW SOLUTIONS #7

(1) Recall how we derived the GCE probability distribution based on the maximization of the entropy S under the constraint of fixed average energy E and particle number N.

(a) Show that one obtains the Bose-Einstein distribution $n_\alpha = \left[e^{\beta(\varepsilon_\alpha - \mu)} - 1\right]^{-1}$ if one extremizes the entropy function

$$S = -k_{\rm B} \sum_\alpha \left[n_\alpha \ln n_\alpha - (1+n_\alpha)\ln(1+n_\alpha)\right]$$

subject to fixed average E and N.

(b) Show that one obtains the Fermi-Dirac distribution $n_\alpha = \left[e^{\beta(\varepsilon_\alpha - \mu)} + 1\right]^{-1}$ if one extremizes the entropy function

$$S = -k_{\rm B} \sum_\alpha \left[n_\alpha \ln n_\alpha + (1-n_\alpha)\ln(1-n_\alpha)\right]$$

subject to fixed average E and N.

Solution:

(a) The variation of the entropy is

$$\delta S = -k_{\rm B} \sum_\alpha \ln\!\left(\frac{n_\alpha}{1+n_\alpha}\right) \delta n_\alpha$$

We also have

$$\delta N = \sum_\alpha \delta n_\alpha \qquad , \qquad \delta E = \sum_\alpha \varepsilon_\alpha \delta n_\alpha \quad .$$

We then write

$$S^* = S - \lambda_N \Big(\sum_\alpha n_\alpha - N\Big) - \lambda_E \Big(\sum_\alpha \varepsilon_\alpha n_\alpha - E\Big)$$

and compute

$$\delta S^* = -\sum_\alpha \left[k_{\rm B} \ln\!\left(\frac{n_\alpha}{1+n_\alpha}\right) + \lambda_N + \lambda_E\, \varepsilon_\alpha\right] \delta n_\alpha = 0 \quad .$$

Setting

$$\lambda_E = \frac{1}{T} \qquad , \qquad \lambda_N = -\frac{\mu}{T} \quad ,$$

we recover the Bose-Einstein distribution,

$$n_\alpha = \frac{1}{e^{(\varepsilon_\alpha - \mu)/k_{\rm B}T} - 1} \quad .$$

(b) The variation of the entropy is

$$\delta S = -k_{\rm B} \sum_\alpha \ln\!\left(\frac{n_\alpha}{1-n_\alpha}\right) \delta n_\alpha$$

As in the Bose-Einstein case, we have

$$\delta N = \sum_\alpha \delta n_\alpha \qquad , \qquad \delta E = \sum_\alpha \varepsilon_\alpha \delta n_\alpha \quad .$$

We then write

$$S^* = S - \lambda_N \Big(\sum_\alpha n_\alpha - N \Big) - \lambda_E \Big(\sum_\alpha \varepsilon_\alpha n_\alpha - E \Big)$$

and compute

$$\delta S^* = -\sum_\alpha \left[k_{\rm B} \ln\!\left(\frac{n_\alpha}{1-n_\alpha}\right) + \lambda_N + \lambda_E \, \varepsilon_\alpha \right] \delta n_\alpha = 0 \quad .$$

Setting

$$\lambda_E = \frac{1}{T} \qquad , \qquad \lambda_N = -\frac{\mu}{T} \quad ,$$

we recover the Bose-Einstein distribution,

$$n_\alpha = \frac{1}{e^{(\varepsilon_\alpha - \mu)/k_{\rm B}T} + 1} \quad .$$

(2) For a noninteracting quantum system with single particle density of states $g(\varepsilon) = A\,\varepsilon^r$ (with $\varepsilon \geq 0$), find the first three virial coefficients for bosons and for fermions.

Solution :

We have

$$n(T,z) = \sum_{j=1}^{\infty} (\pm 1)^{j-1}\, C_j(T)\, z^j \qquad , \qquad p(T,z) = k_{\rm B}T \sum_{j=1}^{\infty} (\pm 1)^{j-1}\, z^j\, j^{-1}\, C_j(T)\, z^j \quad ,$$

where

$$C_j(T) = \int\limits_{-\infty}^{\infty} \! d\varepsilon \; g(\varepsilon)\, e^{-j\varepsilon/k_{\rm B}T} = A\,\Gamma(r+1) \left(\frac{k_{\rm B}T}{j}\right)^{r+1} \quad .$$

Thus, we have

$$\pm\, n v_T = \sum_{j=1}^{\infty} j^{-(r+1)}\, (\pm z)^j$$

$$\pm\, p v_T / k_{\rm B}T = \sum_{j=1}^{\infty} j^{-(r+2)}\, (\pm z)^j \quad ,$$

where

$$v_T = \frac{1}{A\,\Gamma(r+1)\,(k_{\rm B}T)^{r+1}} \quad .$$

has dimensions of volume. Thus, we let $x=\pm z$, and interrogate `Mathematica`:

```
In[1]=  y = InverseSeries[x + x^2/2^(r+1) + x^3/3^(r+1) + x^4/4^(r+1) + O[x]^5]

In[2]=  w = y + y^2/2^(r+2) + y^3/3^(r+2) + y^4/4^(r+2) + O[y]^5  .
```

The result is

$$p = nk_{\rm B}T\Big[1 + B_2(T)\,n + B_3(T)\,n^2 + \ldots\Big] \quad ,$$

where

$$\begin{aligned}
B_2(T) &= \mp 2^{-2-r}\,v_T \\
B_3(T) &= \Big(2^{-2-2r} - 2\cdot 3^{-2-r}\Big)v_T^2 \\
B_4(T) &= \pm 2^{-4-3r}\,3^{-r}\Big(2^{3+2r} - 5\cdot 3^r - 2^r\,3^{1+r}\Big)v_T^3 \quad .
\end{aligned}$$

(3) Consider a system with single particle density of states $g(\varepsilon) = A\,\varepsilon\,\Theta(\varepsilon)\,\Theta(W-\varepsilon)$, which is linear on the interval $[0,W]$ and vanishes outside this interval. Find the second virial coefficient for both bosons and fermions. Plot your results as a function of dimensionless temperature $t = k_{\rm B}T/W$.

Solution :

We need the integrals

$$\begin{aligned}
C_j(T) &= \int\limits_{-\infty}^{\infty}\!\!d\varepsilon\;g(\varepsilon)\,e^{-j\varepsilon/k_{\rm B}T} = A\int\limits_0^W\!\!d\varepsilon\;\varepsilon\,e^{-j\varepsilon/k_{\rm B}T} \\
&= A\bigg(\frac{k_{\rm B}T}{j}\bigg)^{\!2}\int\limits_0^{jW/k_{\rm B}T}\!\!\!dx\;x\,e^{-x} = A\bigg(\frac{k_{\rm B}T}{j}\bigg)^{\!2}(1+x)\,e^{-x}\Big|_{jW/k_{\rm B}T}^{0} \\
&= \tilde A\,\bigg(\frac{k_{\rm B}T}{jW}\bigg)^{\!2}\left\{1 - \bigg(1+\frac{jW}{k_{\rm B}T}\bigg)e^{-jW/k_{\rm B}T}\right\},
\end{aligned}$$

where $\tilde A = AW^2$. We have that $B_2 = \mp C_2/2C_1^2$. Analyzing the temperature dependence, one finds $C_j(T\to 0) \sim \tilde A\,(k_{\rm B}T/jW)^2$, and $C_j(T\to\infty) = \frac{1}{2}\tilde A$. We plot the fermionic result in for $B_2(T)$ in Fig. 1.

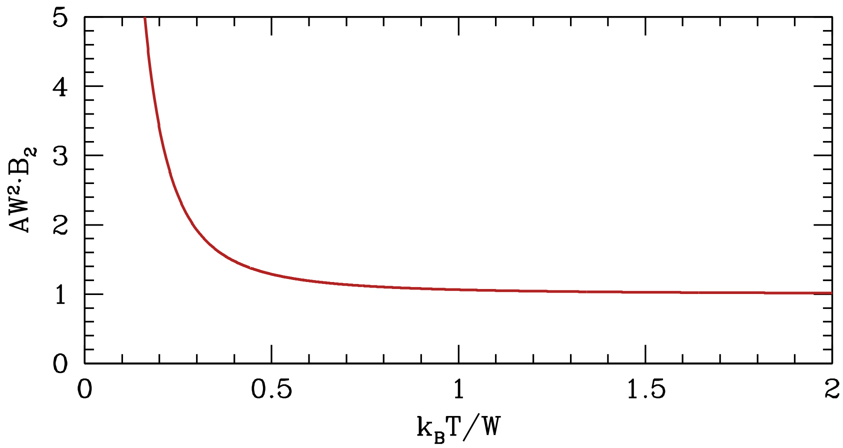

Figure 1: Fermionic second virial coefficient $B_2(T)$ for problem 1.

(4) Using the argument we used in class and in §5.4.3 of the notes, predict the surface temperatures of the remaining planets in our solar system. In each case, compare your answers with the most reliable source you can find. In cases where there are discrepancies, try to come up with a convincing excuse.

Solution :

Relevant planetary data are available from

http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

and from Wikipedia. According to the derivation in the notes, we have

$$T = \left(\frac{R_\odot}{2a}\right)^{1/2} T_\odot \quad ,$$

where $R_\odot = 6.96 \times 10^5$ km and $T_\odot = 5780$ K. From this equation and the reported values for a for each planet, we obtain the following table:
Note that we have included Pluto, because since my childhood Pluto has always been the

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
a (10^8 km)	0.576	1.08	1.50	2.28	7.78	14.3	28.7	45.0	59.1
$T^{\text{obs}}_{\text{surf}}$ (K)	340*	$735^\dagger$	288	210	112	84	53	55	44
$T^{\text{pred}}_{\text{surf}}$ (K)	448	327	278	226	122	89.1	63.6	50.8	44.3

Table 1: Planetary data from GSU web site and from Wikipedia. Observed temperatures are averages except for Mercury (mean equatorial temperature) and Venus (mean temperature below cloud cover).

ninth planet to me. We see that our simple formula works out quite well except for Mercury and Venus. Mercury, being so close to the sun, has enormous temperature fluctuations as a function of location. Venus has a whopping greenhouse effect.