PHYSICS 140A : STATISTICAL PHYSICS
HW SOLUTIONS #1

(1) For each of the following differentials, determine whether it is exact or inexact. If it is exact, find the function whose differential it represents.

(a) $xy^2\,dx + x^2y\,dy$

(b) $z\,dx + x\,dy + y\,dz$

(c) $x^{-2}\,dx - 2x^{-3}\,dy$

(d) $e^x\,dx + \ln(y)\,dy$

Solution :

We will represent each differential as $đA = \sum_\mu A_\mu\,dx^\mu$.

(a) $A_x = xy^2$ and $A_y = x^2y$, so $\frac{\partial A_x}{\partial y} = 2xy = \frac{\partial A_y}{\partial x}$. The differential is exact, and is dA, where $A(x,y) = \frac{1}{2}x^2y^2 + C$, where C is a constant.

(b) With $A_x = z$, $A_y = x$, and $A_z = y$, we have $\frac{\partial A_x}{\partial y} = 0$ and $\frac{\partial A_y}{\partial x} = 1$, as well as $\frac{\partial A_x}{\partial z} = 1$ while $\frac{\partial A_z}{\partial x} = 0$. So the differential is inexact.

(c) $A_x = x^{-2}$ and $A_y = -2x^{-3}$, so $\frac{\partial A_x}{\partial y} = 0$ and $\frac{\partial A_y}{\partial x} = 6x^{-4}$, so the differential is inexact.

(d) $A_x = e^x$ and $A_y = \ln y$, so $\frac{\partial A_x}{\partial y} = 0 = \frac{\partial A_y}{\partial x} = 0$. The differential is exact, with $A(x,y) = e^x + y\ln y - y + C$.

2) Consider an engine cycle which follows the thermodynamic path in Fig. 1. The work material is ν moles of a diatomic ideal gas. BC is an isobar ($dp = 0$), CA is an isochore ($dV = 0$), and along AB one has

$$p(V) = p_{\mathsf{B}} + (p_{\mathsf{A}} - p_{\mathsf{B}}) \cdot \sqrt{\frac{V_{\mathsf{B}} - V}{V_{\mathsf{B}} - V_{\mathsf{A}}}}\,.$$

(a) Find the heat acquired Q_{AB} and the work done W_{AB}.

(b) Find the heat acquired Q_{BC} and the work done W_{BC}.

(c) Find the heat acquired Q_{CA} and the work done W_{CA}.

(d) Find the work W done per cycle.

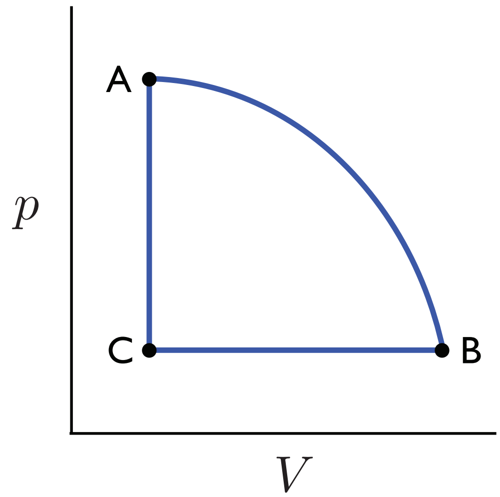

Figure 1: Thermodynamic path for problem 2.

Solution :

Note that $p_\mathsf{C} = p_\mathsf{B}$ and $V_\mathsf{C} = V_\mathsf{A}$, so we will only need to use $\{p_\mathsf{A}, p_\mathsf{B}, V_\mathsf{A}, V_\mathsf{B}\}$ in our analysis. For a diatomic ideal gas, $E = \frac{5}{2}pV$.

(a) We first compute the work done along AB. Let's define u such that $V = V_\mathsf{A} + (V_\mathsf{B} - V_\mathsf{A})\,u$. Then along AB we have $p = p_\mathsf{B} + (p_\mathsf{A} - p_\mathsf{B})\sqrt{1-u}$, and

$$\begin{aligned}
W_\mathsf{AB} &= \int\limits_\mathsf{A}^\mathsf{B} dV\, p \\
&= (V_\mathsf{B} - V_\mathsf{A}) \int\limits_0^1 du\, \Big\{ p_\mathsf{B} + (p_\mathsf{A} - p_\mathsf{B})\sqrt{1-u} \Big\} \\
&= p_\mathsf{B}(V_\mathsf{B} - V_\mathsf{A}) + \tfrac{2}{3}(V_\mathsf{B} - V_\mathsf{A})(p_\mathsf{A} - p_\mathsf{B}) \ .
\end{aligned}$$

The change in energy along AB is

$$(\Delta E)_\mathsf{AB} = E_\mathsf{B} - E_\mathsf{A} = \tfrac{5}{2}(p_\mathsf{B}V_\mathsf{B} - p_\mathsf{A}V_\mathsf{A}) \ ,$$

hence

$$\begin{aligned}
Q_\mathsf{AB} &= (\Delta E)_\mathsf{AB} + W_\mathsf{AB} \\
&= \tfrac{17}{6}\, p_\mathsf{B}V_\mathsf{B} - \tfrac{19}{6}\, p_\mathsf{A}V_\mathsf{A} + \tfrac{2}{3}\, p_\mathsf{A}V_\mathsf{B} - \tfrac{1}{3}\, p_\mathsf{B}V_\mathsf{A} \ .
\end{aligned}$$

(b) Along BC we have

$$W_{\mathsf{BC}} = p_{\mathsf{B}}(V_{\mathsf{A}} - V_{\mathsf{B}})$$

$$(\Delta E)_{\mathsf{BC}} = \tfrac{5}{2} p_{\mathsf{B}}(V_{\mathsf{A}} - V_{\mathsf{B}})$$

$$Q_{\mathsf{BC}} = (\Delta E)_{\mathsf{BC}} + W_{\mathsf{BC}} = \tfrac{7}{2}\, p_{\mathsf{B}}(V_{\mathsf{A}} - V_{\mathsf{B}}) \ .$$

(c) Along CA we have

$$W_{\mathsf{CA}} = 0$$

$$(\Delta E)_{\mathsf{CA}} = \tfrac{5}{2}(p_{\mathsf{A}} - p_{\mathsf{B}})V_{\mathsf{A}}$$

$$Q_{\mathsf{CA}} = (\Delta E)_{\mathsf{CA}} + W_{\mathsf{CA}} = \tfrac{5}{2}(p_{\mathsf{A}} - p_{\mathsf{B}})V_{\mathsf{A}} \ .$$

(d) The work done per cycle is

$$\begin{aligned} W &= W_{\mathsf{AB}} + W_{\mathsf{BC}} + W_{\mathsf{CA}} \\ &= \tfrac{2}{3}(V_{\mathsf{B}} - V_{\mathsf{A}})(p_{\mathsf{A}} - p_{\mathsf{B}}) \ . \end{aligned}$$

(3) $\nu = 8$ moles of a diatomic ideal gas are subjected to a cyclic quasistatic process, the thermodynamic path for which is an ellipse in the (V, p) plane. The center of the ellipse lies at $(V_0, p_0) = (0.25\,\mathrm{m}^3, 1.0\,\mathrm{bar})$. The semimajor axes of the ellipse are $\Delta V = 0.10\,\mathrm{m}^3$ and $\Delta p = 0.20\,\mathrm{bar}$.

(a) What is the temperature at $(V, p) = (V_0 + \Delta V, p_0)$?

(b) Compute the net work per cycle done by the gas.

(c) Compute the internal energy difference $E(V_0 - \Delta V, p_0) - E(V_0, p_0 - \Delta p)$.

(d) Compute the heat Q absorbed by the gas along the upper half of the cycle.

Solution :

(a) The temperature is $T = pV/\nu R$. With $V = V_0 + \Delta V = 0.35\,\mathrm{m}^3$ and $p = p_0 = 1.0\,\mathrm{bar}$, we have

$$T = \frac{(10^5\,\mathrm{Pa})(0.35\,\mathrm{m}^3)}{(8\,\mathrm{mol})(8.31\,\mathrm{J/mol\,K})} = 530\,\mathrm{K} \ .$$

(b) The area of an ellipse is π times the product of the semimajor axis lengths.

$$\oint p\,dV = \pi\,(\Delta p)(\Delta V) = \pi\,(0.20\times 10^6\,\text{bar})\,(0.10\,\text{m}^3) = 6.3\,\text{kJ}\ .$$

(c) For a diatomic ideal gas, $E = \frac{5}{2}pV$. Thus,

$$\Delta E = \tfrac{5}{2}\big(V_0\,\Delta p - p_0\,\Delta V\big) = \tfrac{5}{2}\,(-0.05\times 10^5\,\text{J}) = -13\,\text{kJ}\ .$$

(d) We have $Q = \Delta E + W$, with

$$W = 2\,p_0\,\Delta V + \tfrac{\pi}{2}(\Delta p)(\Delta V) = 23\,\text{kJ}\ ,$$

which is the total area under the top half of the ellipse. The difference in energy is given by $\Delta E = \frac{5}{2}\,p_0\cdot 2\Delta V = 5\,p_0\,\Delta V$, so

$$Q = \Delta E + W = 7\,p_0\,\Delta V + \tfrac{\pi}{2}(\Delta p)(\Delta V) = 73\,\text{kJ}\ .$$

(4) A gas obeys the thermodynamic relation $E(T,V,N) = aNT$ and the equation of state $p = bN^2T/V^2$ where a and b are constants.

(a) What is the isothermal compressibility $\kappa_T = -V^{-1}(\partial V/\partial p)_{T,N}$?

(b) What is the adiabatic equation of state in terms of T, V, and N?

(c) A container of volume V_0 contains N particles of this gas at an initial temperature T_0. The container is opened and the gas expands adiabatically to a volume $V_1 = 2V_0$. Compute the final temperature T_1.

Solution :

(a) We have $V = N(bT)^{1/2}p^{-1/2}$ and thus

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_N = \frac{N}{2V}\,\frac{(bT)^{1/2}}{p^{3/2}} = \frac{1}{2p}\quad .$$

(b) We have $đQ = C_V\,dT + p\,dV = 0$ with $C_V = (\partial E/\partial T)_{V,N} = aN$. Thus

$$đQ = aN\,dT + \frac{bN^2}{V^2}\,T\,dV = 0\quad ,$$

and dividing by bN^2 we obtain

$$\frac{dT}{T} - \frac{b}{a}\, d\left(\frac{N}{V}\right) = 0 \quad ,$$

with N held constant. Integrating, we have

$$\log T = \frac{bN}{aV} + \text{const.} \quad .$$

(c) Setting $V_1 = 2V_0$, we have

$$T_1 = T_0 \exp(-Nb/aV_0) \quad .$$