PHYSICS 140A : STATISTICAL PHYSICS
MIDTERM EXAM SOLUTIONS

(1) ν moles of an ideal diatomic gas are used as the working material for a reversible engine whose cycle ABCDA is depicted in fig. 1. Segments AB and CD are isotherms at temperatures T_2 and T_1 , while segments BC and DA are isochores at volumes V_2 and V_1.

Important: Express all your answers in terms of ν, T_1, T_2, V_1, and V_2 (and not either of the pressures p_1 or p_2).

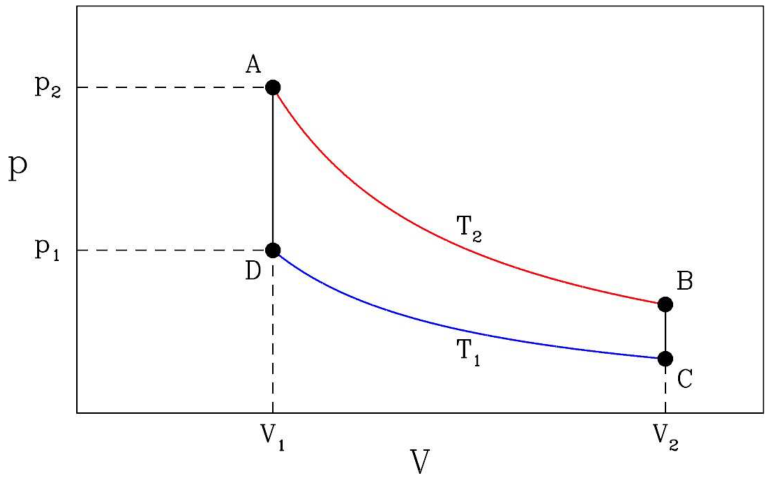

Figure 1: The engine cycle.

(a) What is the work W_{AB} done by the engine and the heat Q_{AB} absorbed by the engine on segment AB? `[10 points]`

(b) What is the work W_{BC} done by the engine and the heat Q_{BC} absorbed by the engine on segment BC? `[10 points]`

(c) What is the work W_{CD} done by the engine and the heat Q_{CD} absorbed by the engine on segment CD? `[10 points]`

(d) What is the work W_{DA} done by the engine and the heat Q_{DA} absorbed by the engine on segment DA? `[10 points]`

(e) What is the efficiency $\eta = W_{\mathsf{cyc}}/Q_{\mathrm{AB}}$? `[10 points]`

Solution :

Since $p = \nu RT/V$ we have:

(a) AB isothermal, so $\Delta E_{\mathrm{AB}} = 0$. Thus $Q_{\mathrm{AB}} = W_{\mathrm{AB}} = \int\limits_{V_1}^{V_2} dV\, p(V) = \nu R T_2 \log(V_2/V_1)$.

(b) BC isochoric, so $W_{\mathrm{BC}} = 0$. Thus $Q_{\mathrm{BC}} = \Delta E_{\mathrm{BC}} = \frac{5}{2}\nu R(T_1 - T_2)$.

(c) CD isothermal, so $\Delta E_{\rm CD} = 0$. Thus $Q_{\rm CD} = W_{\rm CD} = \int\limits_{V_2}^{V_1} dV\, p(V) = \nu R T_1 \log(V_1/V_2)$.

(d) DA isochoric, so $W_{\rm DA} = 0$. Thus $Q_{\rm DA} = \Delta E_{\rm DA} = \frac{5}{2}\nu R(T_2 - T_1)$.

(e) The work done per cycle is $W_{\textsf{cyc}} = W_{\rm AB} + W_{\rm BC} + W_{\rm CD} + W_{\rm DA} = \nu R(T_2 - T_1)\log(V_2/V_1)$. The efficiency is $\eta = W_{\textsf{cyc}}/Q_{\rm AB} = 1 - \frac{T_1}{T_2}$, which is the same as the Carnot cycle efficiency.

(2) Consider the equation of state for a monatomic nonideal gas,

$$\left(p + \frac{\sigma N^2}{V^2}\, k_{\rm B} T\right)(V - N\omega) = N k_{\rm B} T \quad ,$$

where σ and ω are constants. (Note that this is not quite the van der Waals equation of state.)

(a) What are the dimensions of σ and ω? `[10 points]`

(b) Show that the equation of state can be rearranged in the form $p(T,V,N) = k_{\rm B} T \phi(V/N)$, and find the function $\phi(u)$ (where $u = V/N$). `[15 points]`

(c) Find $E(T,V,N)$. *Hint: Consider the low density limit after you ascertain the volume dependence.* `[15 points]`

(d) Find $S(E,V,N)$. `[10 points]`

(e) N atoms of this gas undergo adiabatic free expansion from an initial volume $V_1 = 5N\omega$ to a final volume $V_2 = 2V_1 = 10N\omega$. Find ΔS. `[100 quatloos extra credit]`

Solution :

(a) Since $N k_{\rm B} T/V$ has units of pressure, $\sigma N^2/V$ must be dimensionless. Thus $[\sigma] = {\rm L}$ (volume). Obviously $[\omega] = {\rm L}$ as well.

(b) We rearrange the equation of state to obtain

$$p = \frac{N k_{\rm B} T}{V - N\omega} - \frac{\sigma N^2}{V^2}\, k_{\rm B} T \equiv k_{\rm B} T \phi(V/N) \quad ,$$

where

$$\phi(u) = \frac{1}{u - \omega} - \frac{\sigma}{u^2} \quad .$$

(c) Recall

$$\left(\frac{\partial E}{\partial V}\right)_{T,N} = T\left(\frac{\partial p}{\partial T}\right)_{V,N} - p$$

Thus for our system, E/N must be an intensive function of T alone, independent of volume. Hence the result cannot depend on density either and considering the low density limit, the energy must be that of a monatomic ideal gas, $E(T,V,N)=\frac{3}{2}Nk_{\rm B}T$.

(d) We have

$$dS\big|_N = \frac{dE}{T} + \frac{p}{T}\,dV = \tfrac{1}{2}f\,Nk_{\rm B}\,d\log E + k_{\rm B}\,\phi(V/N)\,dV \quad ,$$

with $f=3$. Integrating, we obtain

$$\begin{aligned} S(E,V,N) &= \tfrac{1}{2}fNk_{\rm B}\,\log E + Nk_{\rm B}\int^{V/N}\!du\,\phi(u) \\ &= \tfrac{1}{2}fNk_{\rm B}\log E + Nk_{\rm B}\log(V-N\omega) + \frac{\sigma N^2 k_{\rm B}}{V} + \chi(N) \quad , \end{aligned}$$

where $\chi(N)$ is a function of N alone. Since $S(E,V,N)$ must be extensive and have dimensions of $k_{\rm B}$, we can write

$$S(E,V,N) = \tfrac{1}{2}fNk_{\rm B}\,\log\!\left(\frac{E}{N\varepsilon_0}\right) + Nk_{\rm B}\log\!\left(\frac{V}{N\omega}-1\right) + \frac{\sigma N^2 k_{\rm B}}{V} + N s_0 \quad ,$$

where ε_0 is a constant with dimensions of energy and s_0 is a constant with dimensions of entropy. (You weren't expected to put the answer in this form.)

(e) We have

$$S(E,2V,N) = S(E,V,N) + Nk_{\rm B}\log\!\left(\frac{2V-N\omega}{V-N\omega}\right) - \frac{\sigma N^2 k_{\rm B}}{2V} = Nk_{\rm B}\left[2\log\tfrac{3}{2} - \frac{\sigma}{10\omega}\right] \quad .$$