# Linear Response and the Meissner Effect

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## 1 Linear Response Theory

### 1.1 Forced damped oscillator

We start with the extremely simple problem of potential energy minimization. Let the potential energy be $V(x) = \frac{1}{2}kx^2$. The minimum lies at $x_{\rm min} = 0$. Now add in an external force so the new potential is

$$\widetilde{V}(x) = \tfrac{1}{2}kx^2 - Fx = \tfrac{1}{2}k\Big(x - \frac{F}{k}\Big)^2 - \frac{F^2}{2k} \quad . \tag{1}$$

The new minimum lies at $x_{\rm min}(F) = F/k$, which is the ratio of the external force $F$ to the spring stiffness $k$. We can also define $\alpha \equiv 1/k$ to be the *susceptibility* of the equilibrium position to the applied force. When the spring is stiff, the susceptibility is low.

Next consider a damped harmonic oscillator subjected to a time-dependent forcing:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \quad , \tag{2}$$

where $\gamma$ is the damping rate ($\gamma > 0$) and $\omega_0$ is the natural frequency in the absence of damping[1]. We adopt the following convention for the Fourier transform of a function $H(t)$:

$$H(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{H}(\omega)\, e^{-i\omega t} \qquad , \qquad \hat{H}(\omega) = \int\limits_{-\infty}^{\infty} dt\, H(t)\, e^{+i\omega t} \quad . \tag{3}$$

Note that if $H(t)$ is a real function, then $\hat{H}(-\omega) = \hat{H}^*(\omega)$. In Fourier space, then, eqn. (2) becomes

$$(\omega_0^2 - 2i\gamma\omega - \omega^2)\, \hat{x}(\omega) = \hat{f}(\omega) \quad , \tag{4}$$

with the solution

$$\hat{x}(\omega) = \frac{\hat{f}(\omega)}{\omega_0^2 - 2i\gamma\omega - \omega^2} \equiv \hat{K}(\omega)\, \hat{f}(\omega) \tag{5}$$

where $\hat{K}(\omega)$ is the *susceptibility* or *response function*:

$$\hat{K}(\omega) = \frac{1}{\omega_0^2 - 2i\gamma\omega - \omega^2} = \frac{-1}{(\omega - \omega_+)(\omega - \omega_-)} \quad , \tag{6}$$

[1] Note that $f(t)$ has dimensions of acceleration.

with $\omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$. The complete solution to (2) is then

$$x(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \frac{\hat{f}(\omega)\, e^{-i\omega t}}{\omega_0^2 - 2i\gamma\omega - \omega^2} + x_{\rm h}(t) \tag{7}$$

where $x_{\rm h}(t)$ is the solution to the homogeneous equation $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$, which is given by $x_{\rm h}(t) = A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t}$. Since $\mathsf{Im}\,(\omega_\pm) < 0$, $x_{\rm h}(t)$ is a *transient* which decays exponentially in time. The coefficients $A_\pm$ may be chosen to satisfy initial conditions on $x(0)$ and $\dot{x}(0)$, but the system 'loses its memory' of these initial conditions after a finite time, and in steady state all that is left is the inhomogeneous piece, which is completely determined by the forcing.

In the time domain, we can write

$$x(t) = \int\limits_{-\infty}^{\infty} dt' \; K(t-t')\, f(t') \qquad , \qquad K(s) \equiv \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \; \hat{K}(\omega)\, e^{-i\omega s} \quad . \tag{8}$$

From the theory of complex integration, we have the following important result:

Claim: The response is *causal*, *i.e.* $K(t-t') = 0$ when $t < t'$, provided that $\hat{K}(\omega)$ is analytic in the upper half plane of the variable $\omega$.

Proof: Consider eqn. (8). Of $\hat{K}(\omega)$ is analytic in the upper half plane, then closing in the UHP we obtain $K(s<0) = 0$.

Formally we may write

$$K(t-t') = \frac{\delta x(t)}{\delta f(t')} \quad , \tag{9}$$

which is to say $K(t-t')$ is the *functional derivative* of $x(t)$ with respect to $f(t')$. It tells us how they system responds at time $t$ to a variation in the forcing at a time $t'$. Due to causality, there is no response if $t' > t$ since the future is no influence on the present.

### 1.2 Quantum systems

A superconductor is a quantum many-body system describing a collection of electrons and ions. You know that quantum systems are described by a Hamiltonian operator $\hat{H}_0$, which we assume to be time-independent. $\hat{H}_0$ has an eigenspectrum, *viz.* $\hat{H}_0 | \, \Psi_\alpha \, \rangle = E_\alpha \, | \, \Psi_\alpha \, \rangle$. The eigenstate $| \, \Psi_0 \, \rangle$ corresponding to the lowest energy eigenvalue $E_0$ is the *ground state* of the system, and its properties dominate as $T \to 0$.

In linear response, we are concerned with perturbing the Hamiltonian with a term $\hat{V}(t)$ which in general is time-dependent. Thus we write $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$, where

$$\hat{V}(t) = -\sum_i \hat{Q}_i\,\phi_i(t) \qquad \text{or} \qquad \hat{V}(t) = -\sum_i \int d^d x\, \hat{Q}_i(\boldsymbol{x})\,\phi_i(\boldsymbol{x},t) \quad . \tag{10}$$

Here each $\hat{Q}_i$ is an operator labeled by an index $i$, and possibly a spatial index $\boldsymbol{x}$ as well for continuous systems. The quantities $\phi_i(t)$ or $\phi_i(\boldsymbol{x},t)$ are spatiotemporally varying fields, such as a local scalar potential or local magnetic field. Examples:

$$\hat{V}(t) = \begin{cases} -\hat{\boldsymbol{M}}\cdot\boldsymbol{B}(t) & \text{magnetic moment – magnetic field} \\ +\int d^3x\,\hat{\varrho}(\boldsymbol{x})\,\phi(\boldsymbol{x},t) & \text{charge density – scalar potential} \\ -\frac{1}{c}\int d^3x\,\hat{\boldsymbol{\jmath}}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x},t) & \text{electromagnetic current – vector potential} \end{cases} \tag{11}$$

What we are after is the time-dependent expectation value of $\hat{Q}_i$ (suppressing any spatial indices $\boldsymbol{x}$ for now),

$$Q_i(t) \equiv \langle\, \Psi(t)\,|\,\hat{Q}_i\,|\,\Psi(t)\,\rangle \tag{12}$$

where $i\hbar\partial_t\,|\Psi(t)\rangle = \hat{H}(t)\,|\Psi(t)\rangle$. Without loss of generality, we may assume that the operators $\{\hat{Q}_i\}$ are each defined in such a way that $\langle\Psi_0|\,\hat{Q}_i\,|\Psi_0\rangle = 0$ for all $i$, where $|\Psi_0\rangle$ is the ground state of $H_0$, which itself is in general an interacting many-body Hamiltonian. We therefore expect that, to lowest nontrivial order in the fields $\{\phi_i\}$, that the observed response should be linear, *i.e.*

$$Q_i(t) = \int\limits_{-\infty}^{\infty} dt'\, K_{ij}(t-t')\,\phi_j(t') + \mathcal{O}(\phi^2) \quad . \tag{13}$$

$K_{ij}(t-t')$ is the quantum mechanical response function, describing how $\langle\hat{Q}_i(t)\rangle$ responds at time $t$ to a change in the applied field $\phi_j(t')$ at time $t'$. We'll skip the derivation, which makes use of QM first order perturbation theory, but one important similarity with the classical cases is that the responses are all causal, *i.e.* $K_{ij}(t-t') = 0$ for $t < t'$.

For completeness, the formal result is

$$K_{ij}(t-t') = \frac{i}{\hbar}\,\big\langle\big[\hat{Q}_i(t), \hat{Q}_j(t')\big]\big\rangle\,\Theta(t-t') \quad , \tag{14}$$

where $\Theta(s)$ is the step function such that $\Theta(s>0) = 0$, ergo causality. The main part here is the expectation of the commutator of the $\hat{Q}$ operators in the "interaction representation",

$$\hat{Q}_j(t) \equiv e^{i\hat{H}_0 t/\hbar}\,\hat{Q}_j\,e^{-i\hat{H}_0 t/\hbar} \quad . \tag{15}$$

The expectation is taken in the ground state or in a thermal ensemble of Hamiltonian eigenstates of $\hat{H}_0$. You will learn about this stuff in Physics 130ABC.

Note that eqn. 13 includes terms in the field beyond linear order. Such nonlinearities did not enter into our analysis of the forced damped harmonic oscillator because that was intrinsically a linear system. In the real world, there are always nonlinearities, however these are negligible for sufficiently small values of the applied field.

## 2 Electromagnetic Response

Finally we come to the main event: what happens when a magnetic field is imposed on a superconductor. We will consider this problem in the context of linear response. I stress that there are some very interesting *nonlinear* consequences of magnetic fields in superconductors, but here we are interested only in linear response. Consider an interacting system consisting of electrons of charge $-e$ in the presence of a time-varying electromagnetic field. The electromagnetic field is given in terms of the 4-potential $A^\mu = (A^0, \boldsymbol{A})$:

$$\boldsymbol{E} = -\boldsymbol{\nabla} A^0 - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \qquad , \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad . \tag{16}$$

The electromagnetic response tensor $K_{\mu\nu}$ is defined via the linear response relation,

$$\big\langle j_\mu(\boldsymbol{x},t)\big\rangle = -\frac{c}{4\pi}\int\! dt' \!\int\! d^3x'\; K_{\mu\nu}(\boldsymbol{x}t;\boldsymbol{x}'t')\, A^\nu(\boldsymbol{x}',t') \quad , \tag{17}$$

which is valid to first order in the external 4-potential $A^\mu$. Assuming $H_0$ is space and time translation invariant, $K_{\mu\nu}(\boldsymbol{x}t;\boldsymbol{x}'t') = K_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}', t-t')$ is a function only of the spatiotemporal separations. We may adopt a gauge in which $A^0 = 0$, $\boldsymbol{E} = -c^{-1}\dot{\boldsymbol{A}}$, and $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. To satisfy Maxwell's equations[2], we have $\boldsymbol{q}\cdot\boldsymbol{A}(\boldsymbol{q},\omega) = 0$, *i.e.* $\boldsymbol{A}(\boldsymbol{q},\omega)$ is purely transverse. We then have

$$\big\langle \boldsymbol{j}(\boldsymbol{q},\omega)\big\rangle = -\frac{c}{4\pi}\, K_\perp(\boldsymbol{q},\omega)\, \boldsymbol{A}(\boldsymbol{q},\omega) \quad . \tag{18}$$

Here we have shifted from $(\boldsymbol{x},t)$ to Fourier space variables $(\boldsymbol{q},\omega)$ by Fourier transforming with respect to $\boldsymbol{x}-\boldsymbol{x}'$ and $t-t'$. The quantity $\boldsymbol{j}(\boldsymbol{x},t)$ is the charge current operator, and $K_\perp$ is a particular combination of the spatial components $K_{ij}$ of the electromagnetic response tensor[3]. From dimensional analysis, we find $\big[K_\perp(\boldsymbol{q},\omega)\big] = \mathrm{L}^{-2}$, *i.e.* inverse area.

Eqn. 18 describes the Meissner effect. To see this, we first take the static limit in which the applied field $\boldsymbol{B}(\boldsymbol{x},t)$ and its associated vector potential are time-independent. We are then interested in the response at $\omega = 0$. If we assume that $K_\perp(\boldsymbol{q},\omega\to 0) = \lambda_{\mathrm{L}}^{-2}$ for all $\boldsymbol{q}$, then

$$\nabla^2 \boldsymbol{A} = -\boldsymbol{\nabla}\times\boldsymbol{B} = -\frac{4\pi}{c}\langle \boldsymbol{j}\rangle = \lambda_{\mathrm{L}}^{-2}\boldsymbol{A} \quad . \tag{19}$$

This tells us that the vector potential $\boldsymbol{A}(\boldsymbol{x})$ and the field $\boldsymbol{B}(\boldsymbol{x})$ decay on a length scale $\lambda_{\mathrm{L}}$ inside the system. A superconductor is a state of matter for which $\lambda_{\mathrm{L}} \neq 0$. In a normal metal $\lambda_{\mathrm{L}} = \infty$, *i.e.* $\lim_{\boldsymbol{q}\to 0} K_\perp(\boldsymbol{q},0) = 0$.

Within London theory, the assumption that $K_\perp(\boldsymbol{q},\omega\to 0) = \lambda_{\mathrm{L}}^{-2}$ for all $\boldsymbol{q}$ pertains. But this is a rather severe approximation and in general one must account for the $\boldsymbol{q}$-dependence of $K_\perp(\boldsymbol{q},\omega = 0)$. (In an isotropic system, the dependence is only on the magnitude $q = |\boldsymbol{q}|$

[2] *I.e.* if $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$, which pertains when there is local charge neutrality.

[3] One has $K_{ij}(\boldsymbol{q},\omega) = K_{\parallel}(\boldsymbol{q},\omega)\,\hat{q}_i\,\hat{q}_j + K_\perp(\boldsymbol{q},\omega)\,\big(\delta_{ij} - \hat{q}_i\,\hat{q}_j\big)$, where $\hat{q}_i \equiv q_i/|\boldsymbol{q}|$.

of the wavevector.) Since the function $K_{\perp}(\boldsymbol{q},0)$ has dimensions of $L^{-2}$, assuming spatial isotropy, we may define another length scale, $\lambda$, by

$$\lambda = \frac{2}{\pi}\int\limits_0^{\infty} \frac{dq}{q^2 + K_{\perp}(q,0)} \quad . \tag{20}$$

This is sometimes called the *true penetration depth.* Note that if $K_{\perp}(q,0)$ is $q$-independent then $\lambda = \lambda_{\rm L}$. At finite temperature, the EM response kernel $K_{\mu\nu}(\boldsymbol{q},\omega)$ and hence the penetration depth are temperature dependent: $\lambda = \lambda(T)$. Computing $K_{\mu\nu}$ within BCS theory, one obtains[4]

$$\frac{\lambda(T)}{\lambda_0} = \left[\frac{\Delta_0}{\Delta(T)}\,\mathrm{ctnh}\left(\frac{\Delta(T)}{2k_{\rm B}T}\right)\right]^{1/2} \quad , \tag{21}$$

where $\Delta(T)$ is the temperature-dependent BCS energy gap, $\Delta_0 = \Delta(0)$, and

$$\lambda_0 = \frac{8\cdot 3^{1/6}}{9\cdot(2\pi)^{1/3}}\left[\xi_0\,\lambda_{\rm L}^2(0)\right]^{1/3} \quad , \tag{22}$$

with $\xi_0$ the BCS coherence length.

[4]See Fetter and Walecka, §52.