PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #2 SOLUTIONS

(1) Turkey typically cooks at a temperature of $350^\circ\,\mathrm{F}$. Calculate the total electromagnetic energy inside an over of volume $V = 1.0\,\mathrm{m}^3$ at this temperature. Compare it to the thermal energy of the air in the oven at the same temperature.

The total electromagnetic energy is

$$E = 3pV = \frac{\pi^2}{15}\,\frac{V\,(k_{\mathrm{B}}T)^4}{(\hbar c)^3} = 3.78 \times 10^{-5}\,\mathrm{J} \quad .$$

For air, which is a diatomic ideal gas, we have $E = \frac{5}{2}pV$. What do we take for p? If we assume that oven door is closed at an initial temperature of $63^\circ\,\mathrm{F}$ which is $300\,\mathrm{K}$, then with a final temperature of $350^\circ\,\mathrm{F} = 450\,\mathrm{K}$, we have an increase in the absolute temperature by 50%, hence a corresponding pressure increase of 50%. So we set $p = \frac{3}{2}$ atm and we have

$$E = \tfrac{5}{2} \cdot \tfrac{3}{2}(1.013 \times 10^5\,\mathrm{Pa})(1.0\,\mathrm{m}^3) = 3.80 \times 10^5\,\mathrm{J} \quad ,$$

which is about ten orders of magnitude larger.

(2) Let L denote the number of single particle energy levels and N the total number of particles for a given system. Find the number of possible N-particle states $\Omega(L,N)$ for each of the following situations:

(a) Distinguishable particles with $L = 3$ and $N = 3$.

$\Omega_{\mathrm{D}}(3,3) = 3^3 = 27$.

(b) Bosons with $L = 3$ and $N = 3$.

$\Omega_{\mathrm{BE}}(3,3) = \binom{5}{3} = 10$.

(c) Fermions with $L = 10$ and $N = 3$.

$\Omega_{\mathrm{FD}}(10,3) = \binom{10}{3} = 120$.

(d) Find a general formula for $\Omega_{\mathrm{D}}(L,N)$, $\Omega_{\mathrm{BE}}(L,N)$, and $\Omega_{\mathrm{FD}}(L,N)$.

The general results are

$$\Omega_{\mathrm{D}}(L,N) = L^N \quad , \quad \Omega_{\mathrm{BE}}(L,N) = \binom{N+L-1}{N} \quad , \quad \Omega_{\mathrm{FD}}(L,N) = \binom{L}{N} \quad .$$

(3) A species of noninteracting quantum particles in $d = 2$ dimensions has dispersion $\varepsilon(\boldsymbol{k}) = \varepsilon_0 |\boldsymbol{k}\ell|^{3/2}$, where ε_0 is an energy scale and ℓ a length.

(a) Assuming the particles are $S = 0$ bosons obeying photon statistics, compute the heat capacity C_V.

The density of states is

$$g(\varepsilon) = \frac{1}{2\pi}\,\frac{k}{d\varepsilon/dk} = \frac{k^{1/2}}{3\pi\varepsilon_0\,\ell^{3/2}} = \frac{\varepsilon^{1/3}}{3\pi\ell^2\varepsilon_0^{4/3}} \quad .$$

The total energy is

$$E = A\int_0^\infty d\varepsilon\; g(\varepsilon)\,\frac{\varepsilon}{e^{\varepsilon/k_B T}-1} = \frac{A}{3\pi\ell^2}\,\Gamma\big(\tfrac{7}{3}\big)\,\zeta\big(\tfrac{7}{3}\big)\,\frac{(k_B T)^{7/3}}{\varepsilon_0^{4/3}} \quad ,$$

where A is the system area. Thus,

$$C_A(T) = \bigg(\frac{\partial E}{\partial T}\bigg)_{\!A} = \frac{Ak_B}{3\pi\ell^2}\,\Gamma\big(\tfrac{10}{3}\big)\,\zeta\big(\tfrac{7}{3}\big)\bigg(\frac{k_B T}{\varepsilon_0}\bigg)^{4/3} \quad .$$

(b) Assuming the particles are $S = 0$ bosons, is there an Bose condensation transition? If yes, compute the condensation temperature $T_c(n)$ as a function of the particle density. If no, compute the low-temperature behavior of the chemical potential $\mu(n, T)$.

The following integral may be useful:

$$\int_0^\infty \frac{u^{s-1}\,du}{e^u - 1} = \Gamma(s)\sum_{n=1}^\infty n^{-s} \equiv \Gamma(s)\,\zeta(s) \quad ,$$

where $\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann zeta-function.

The condition for Bose-Einstein condensation is

$$n = \int_0^\infty d\varepsilon\; g(\varepsilon)\,\frac{1}{e^{\varepsilon/k_B T_c}-1} = \frac{1}{3\pi\ell^2}\,\Gamma\big(\tfrac{4}{3}\big)\,\zeta\big(\tfrac{4}{3}\big)\bigg(\frac{k_B T_c}{\varepsilon_0}\bigg)^{4/3} \quad .$$

Thus,

$$T_c = \frac{\varepsilon_0}{k_B}\left(\frac{3\pi\ell^2 n}{\Gamma\big(\frac{4}{3}\big)\,\zeta\big(\frac{4}{3}\big)}\right)^{3/4} \quad .$$

(4) Hydrogen (H_2) freezes at 14 K and boils at 20 K under atmospheric pressure. The density of liquid hydrogen is $70\,\mathrm{kg/m^3}$. Hydrogen molecules are bosons. No evidence has been found for Bose-Einstein condensation of hydrogen. Why not?

If we treat the H_2 molecules as bosons, and we ignore the rotational freedom, which is appropriate at temperatures below $\Theta_{\rm rot} = 85.4\,{\rm K}$, we have

$$T_{\rm c} = \frac{2\pi\hbar^2}{m k_{\rm B}} \left(\frac{n}{\zeta\big(\frac{3}{2}\big)}\right)^{2/3} = 6.1\,{\rm K} \quad .$$

Thus, the critical temperature for ideal gas Bose-Einstein condensation is significantly below the freezing temperature for H_2. The freezing transition into a regular solid preempts any BEC phenomena.

(5) (Difficult) Consider a three-dimensional Bose gas of particles which have two internal polarization states, labeled by $\sigma = \pm 1$. The single particle energies are given by

$$\varepsilon(\boldsymbol{k},\sigma) = \frac{\hbar^2 \boldsymbol{k}^2}{2m} + \sigma\Delta \quad ,$$

where $\Delta > 0$.

(a) Find the density of states per unit volume $g(\varepsilon)$.

Let $g_0(\varepsilon)$ be the DOS per unit volume for the case $\Delta = 0$. Then

$$g_0(\varepsilon)\, d\varepsilon = \frac{d^3\!k}{(2\pi)^3} = \frac{k^2\, dk}{2\pi^2} \quad \Rightarrow \quad g_0(\varepsilon) = \frac{2}{\sqrt{\pi}} \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \varepsilon^{1/2}\, \Theta(\varepsilon) \quad .$$

For finite Δ, the single particle energies are shifted uniformly by $\pm\Delta$ for the $\sigma = \pm 1$ states, hence

$$g(\varepsilon) = g_0(\varepsilon + \Delta) + g_0(\varepsilon - \Delta) \quad .$$

(b) Find an implicit expression for the condensation temperature $T_{\rm c}(n, \Delta)$. When $\Delta \to \infty$, your expression should reduce to the familiar one derived in class.

For Bose statistics, we have in the uncondensed phase,

$$\begin{aligned} n &= \int\limits_{-\infty}^{\infty}\! d\varepsilon\, \frac{g(\varepsilon)}{e^{(\varepsilon-\mu)/k_{\rm B}T} - 1} \\ &= {\rm Li}_{3/2}\big(e^{(\mu+\Delta)/k_B T}\big)\, \lambda_T^{-3} + {\rm Li}_{3/2}\big(e^{(\mu-\Delta)/k_B T}\big)\, \lambda_T^{-3} \quad . \end{aligned}$$

In the condensed phase, $\mu = -\Delta - \mathcal{O}(N^{-1})$ is pinned just below the lowest single particle energy, which occurs for $\boldsymbol{k} = 0$ and $\sigma = -1$. We then have

$$n = n_0 + \zeta(3/2)\, \lambda_T^{-3} + {\rm Li}_{3/2}\big(e^{-2\Delta/k_{\rm B}T}\big)\, \lambda_T^{-3} \quad .$$

To find the critical temperature, set $n_0 = 0$ and $\mu = -\Delta$:

$$n\, \lambda_{T_{\rm c}}^3 = \zeta(3/2) + {\rm Li}_{3/2}\big(e^{-2\Delta/k_{\rm B}T_{\rm c}}\big) \quad .$$

This is a nonlinear and implicit equation for $T_{\rm c}(n,\Delta)$. When $\Delta=\infty$, we have

$$k_{\rm B}T_{\rm c}(n,\infty)=\frac{2\pi\hbar^2}{m}\left(\frac{n}{\zeta(3/2)}\right)^{2/3} \quad .$$

(c) When $\Delta=\infty$, the condensation temperature should agree with the familiar result for three-dimensional Bose condensation. Assuming $\Delta \gg k_{\rm B}T_{\rm c}(n,\Delta=\infty)$, find analytically the leading order difference $\delta T_{\rm c}(n,\Delta)\equiv T_{\rm c}(n,\Delta)-T_{\rm c}(n,\Delta=\infty)$.

For finite Δ, we still have the implicit nonlinear equation to solve, but in the limit $\Delta \gg k_{\rm B}T_{\rm c}$, we can expand $T_{\rm c}(n,\Delta)=T_{\rm c}(n,\infty)+\delta T_{\rm c}(n,\Delta)$. We may then set $T_{\rm c}(n,\Delta)$ to $T_{\rm c}(n,\infty)$ in the second term of our nonlinear implicit equation, move this term to the LHS, whence

$$\left(\frac{T_{\rm c}(n,\Delta)}{T_{\rm c}(n,\infty)}\right)^{3/2}=1-\frac{1}{\zeta(3/2)}\,\mathrm{Li}_{3/2}\big(e^{-2\Delta/k_{\rm B}T_{\rm c}(n,\infty)}\big) \quad ,$$

which is a simple algebraic equation for $T_{\rm c}(n,\Delta)$. The second term on the RHS is tiny since $\Delta \gg k_{\rm B}T_{\rm c}(n,\infty)$. We then find

$$T_{\rm c}(n,\Delta)=T_{\rm c}(n,\infty)\left\{1-\tfrac{2}{3\,\zeta(3/2)}\,e^{-2\Delta/k_{\rm B}T_{\rm c}(n,\infty)}+\mathcal{O}\big(e^{-4\Delta/k_{\rm B}T_{\rm c}(n,\infty)}\big)\right\} \quad ,$$

and thus the shift in the condensation temperature is

$$\delta T_{\rm c}(n,\Delta)=-\frac{2}{3\,\zeta(3/2)}\,e^{-2\Delta/k_{\rm B}T_{\rm c}(n,\infty)}\,T_{\rm c}(n,\infty) \quad .$$

Note that

$$\frac{2\Delta}{k_{\rm B}T_{\rm c}(n,\infty)}=\frac{m\Delta}{\pi\hbar^2}\left(\frac{\zeta(3/2)}{n}\right)^{2/3} \quad .$$