[1] Consider the matrix

\[ M = \begin{pmatrix} 4 & 4 \\ -1 & 9 \end{pmatrix} \]

(a) Find the characteristic polynomial \( P(\lambda) = \det(\lambda I - M) \) and the eigenvalues.

(b) For each eigenvalue \( \lambda_\alpha \), find the associated right eigenvector \( R_\alpha \) and left eigenvector \( L_\alpha \). Normalize your eigenvectors so that \( \langle L_\alpha | R_\beta \rangle = \delta_{\alpha\beta} \).

(c) Show explicitly that \( M_{ij} = \sum_\alpha \lambda_\alpha R_\alpha^i L_\alpha^j \).

[2] Consider a three-state system with the following transition rates:

\[ W_{12} = 0 \quad , \quad W_{21} = \gamma \quad , \quad W_{23} = 0 \quad , \quad W_{32} = 3\gamma \quad , \quad W_{13} = \gamma \quad , \quad W_{31} = \gamma \]

(a) Find the matrix \( \Gamma \) such that \( \dot{P}_i = -\Gamma_{ij} P_j \).

(b) Find the equilibrium distribution \( P_{\text{eq}}^i \).

(c) Does this system satisfy detailed balance? Why or why not?

[3] A Markov chain is a process which describes transitions of a discrete stochastic variable occurring at discrete times. Let \( P_i(t) \) be the probability that the system is in state \( i \) at time \( t \). The evolution equation is

\[ P_i(t+1) = \sum_j Q_{ij} P_j(t) \]

The transition matrix \( Q_{ij} \) satisfies \( \sum_i Q_{ij} = 1 \) so that the total probability \( \sum_i P_i(t) \) is conserved. The element \( Q_{ij} \) is the conditional probability that for the system to evolve to state \( i \) at time \( t+1 \) given that it was in state \( j \) at time \( t \). Now consider a group of Physics graduate students consisting of three theorists and four experimentalists. Within each group, the students are to be regarded as indistinguishable. Together, the students rent two apartments, A and B. Initially the three theorists live in A and the four experimentalists live in B. Each month, a random occupant of A and a random occupant of B exchange domiciles. Compute the transition matrix \( Q_{ij} \) for this Markov chain, and compute the average fraction of the time that B contains two theorists and two experimentalists, averaged over the effectively infinite time it takes the students to get their degrees. \textit{Hint:} \( Q \) is a \( 4 \times 4 \) matrix.

[4] Consider a modified version of the Kac ring model where each spin exists in one of three states: A, B, or C. The flippers rotate the internal states cyclically: A→B→C→A.
(a) What is the Poincaré recurrence time for this system? Hint: the answer depends on whether or not the total number of flippers is a multiple of 3.

(b) Simulate the system numerically. Choose a ring size on the order of \( N = 10,000 \) and investigate a few flipper densities: \( x = 0.001, x = 0.01, x = 0.1, x = 0.99 \). Remember that the flippers are located randomly at the start, but do not move as the spins evolve. Starting from a configuration where all the spins are in the A state, plot the probabilities \( p_A(t), p_B(t), \) and \( p_C(t) \) versus the discrete time coordinate \( t \), with \( t \) ranging from 0 to the recurrence time. If you can, for each value of \( x \), plot the three probabilities in different colors or line characteristics (e.g. solid, dotted, dashed) on the same graph.

(c) Let’s call \( a_t = p_A(t) \), etc. Explain in words why the Stosszahlansatz results in the equations

\[
\begin{align*}
a_{t+1} &= (1 - x) a_t + x c_t \\
b_{t+1} &= (1 - x) b_t + x a_t \\
c_{t+1} &= (1 - x) c_t + x b_t
\end{align*}
\]

This describes what is known as a Markov process, which is governed by coupled equations of the form \( P_i(t + 1) = \sum_j Q_{ij} P_j(t) \), where \( Q \) is the transition matrix. Find the \( 3 \times 3 \) transition matrix for this Markov process.

(d) Show that the total probability is conserved by a Markov process if \( \sum_j Q_{ij} = 1 \) and verify this is the case for the equations in (c).

(e) One can then eliminate \( c_t = 1 - a_t - b_t \) and write these as two coupled equations. Show that if we define

\[
\begin{align*}
\tilde{a}_t &\equiv a_t - \frac{1}{3} \\
\tilde{b}_t &\equiv b_t - \frac{1}{3}
\end{align*}
\]

that we can write

\[
\begin{pmatrix} \tilde{a}_{t+1} \\ \tilde{b}_{t+1} \end{pmatrix} = R \begin{pmatrix} \tilde{a}_t \\ \tilde{b}_t \end{pmatrix}
\]

and find the \( 2 \times 2 \) matrix \( R \). Note that this is not a Markov process in A and B, since total probability for the A and B states is not itself conserved. Show that the eigenvalues of \( R \) form a complex conjugate pair. Find the amplitude and phase of these eigenvalues. Show that the amplitude never exceeds unity.

(f) The fact that the eigenvalues of \( R \) are complex means that the probabilities should oscillate as they decay to their equilibrium values \( p_A = p_B = p_C = \frac{1}{3} \). Can you see this in your simulations?