Physics 2D

Quiz 5 Solutions

Fall 2022

Problem 1

$$\psi(x) = C x e^{-x/2a_0} \quad ; \quad |\psi(x)|^2 = C^2 x^2 e^{-x/a_0}$$

$$\langle x \rangle = \frac{\int_0^\infty dx\, x\, |\psi(x)|^2}{\int_0^\infty dx\, |\psi(x)|^2} \Rightarrow$$

Use

$$\int_0^\infty dx\, x^n e^{-\lambda x} = \frac{n!}{\lambda^{n+1}}$$

$$\langle x \rangle = \frac{\int_0^\infty dx\, x^3 e^{-x/a_0}}{\int_0^\infty dx\, x^2 e^{-x/a_0}} = \frac{3!\, a_0^4}{2!\, a_0^3} = \boxed{3a_0 = \langle x \rangle}$$

Alternatively, compute first C from normalization:

$$1 = \int_0^\infty dx\, |\psi(x)|^2 = C^2 \int_0^\infty dx\, x^2 e^{-x/a_0} = C^2 \cdot 2!\, a_0^3$$

$$\Rightarrow C^2 = \frac{1}{2a_0^3} \Rightarrow C = \frac{1}{(2a_0^3)^{1/2}}$$

Then

$$\langle x \rangle = C^2 \int_0^\infty dx\, x^3 e^{-x/a_0} = \frac{1}{2a_0^3} \cdot 3!\, a_0^4 = 3a_0$$

Problem 2

$l = 3, \quad m_l = 2$

(a) $n = 4, 5, 6, \ldots \quad (n \geq 4, \text{ integer})$

(b) $|\vec{L}| = \sqrt{l(l+1)}\,\hbar = \sqrt{12}\,\hbar$

(c) $\cos\theta = \dfrac{L_z}{|\vec{L}|}$; $L_z = m_l \hbar = 2\hbar$

$\Rightarrow \cos\theta = \dfrac{2}{\sqrt{12}} \Rightarrow \boxed{\theta = 54.7^\circ}$

(d) No uncertainty in L_z. $\boxed{\Delta L_z = 0}$

(e) $\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2}$

$\langle L_x \rangle = 0$ by symmetry, and $\langle L_x^2 \rangle = \langle L_y^2 \rangle$ by symmetry

$\langle L^2 \rangle = l(l+1)\hbar^2 = 2\langle L_x^2 \rangle + \langle L_z^2 \rangle = 2\langle L_x^2 \rangle + m_l^2 \hbar^2 \Rightarrow$

$\langle L_x^2 \rangle = \dfrac{(l(l+1) - m_l^2)\,\hbar^2}{2} = \dfrac{12-4}{2}\hbar^2 = 2^2\hbar^2$

$\Rightarrow \boxed{\Delta L_x = \sqrt{\langle L_x^2 \rangle} = 2\hbar}$

Problem 3

$$R(r) = C r^2 e^{-r/a_0}$$

$$P(r) = r^2 R^2(r) = C^2 r^6 e^{-2r/a_0}$$

$$P'(r_m) = 0 = 6 r_m^5 - \frac{2 r_m^6}{a_0} \Rightarrow \boxed{r_m = 3a_0}$$

In Bohr atom, $r_n = \frac{a_0}{Z} n^2$. So assuming $r_m = r_n$,

$$3a_0 = \frac{a_0}{Z} n^2 \Rightarrow \boxed{Z = \frac{n^2}{3}}$$

So $n = 3$, $\boxed{Z = 3}$ is one possibility. The smallest that satisfies that condition.

We could think $n = 6$, $Z = 12$ is also possible. However there is no radial wave function with $n = 6$ that looks like what is given.

Problem 4

The energy of the 2p level in the absence of magnetic field is

$$E_{2p}^{0} = -\frac{E_0}{4} \qquad \text{with } E_0 = 13.6\,eV$$

In the presence of magnetic field B, ignoring spin,

$$E_{2p} = E_{2p}^{0} + \mu_B B m_e \quad , \quad m_e = \pm 1, 0$$

The transition is to the 1s state, with $E_{1s} = -E_0$

$$\boxed{B = 30T}$$

$$\Rightarrow E_{2p} - E_{1s} = \frac{3}{4} E_0 + \mu_B B m_e = \frac{hc}{\lambda_{m_e}}$$

$$\boxed{\mu_B = 5.79 \times 10^{-5} \frac{eV}{T}}$$

$$\Rightarrow \lambda_{m_e} = \frac{hc}{\frac{3}{4} E_0 + \mu_B B m_e} = \frac{1240}{10.2 + 1.737 \times 10^{-3} m_e} \,nm$$

$$\boxed{\begin{aligned} \lambda_{m_e=1} &= 121.548\,nm \\ \lambda_{m_e=0} &= 121.569\,nm \\ \lambda_{m_e=-1} &= 121.589\,nm \end{aligned}}$$

Problem 5

$$\Delta E = E_o (Z-1)^2 \cdot \frac{3}{4} = \frac{hc}{\lambda} \Rightarrow$$

$$\Rightarrow (Z-1)^2 = \frac{4}{3}\frac{hc}{E_o} \Rightarrow Z = \left(\frac{4}{3}\frac{hc}{E_o}\right)^{1/2} + 1$$

$$\Rightarrow \boxed{Z = \left(\frac{4}{3}\,\frac{1240}{13.6}\right)^{1/2} + 1 = 12} \quad (a)$$

(b) Electronic configuration :

$\boxed{1s^2\, 2s^2\, 2p^6\, 3s^2}$ the element is Mg