\[ |\psi_{2,l,\pm 1}(r, \theta, \phi)|^2 dV = \frac{1}{24a_0^5} \frac{r^2}{a_0^2} e^{-r/a_0} \left( \frac{3}{8\pi} \sin^2 \theta \right) r^2 \sin \theta dr d\theta d\phi \]

For \( \theta = 0 \), both probabilities are zero due to the \( \sin \theta \) terms.

(b) For \( \theta = 90^\circ \), the 2,1,0 probability is zero due to the \( \cos \theta \) term. With \( dr = 0.02a_0 \), \( d\theta = 0.11^\circ = 0.00192 \text{ rad} \), and \( d\phi = 0.25^\circ = 0.00436 \text{ rad} \), the 2,1,\( \pm 1 \) probability is

\[ |\psi_{2,l,\pm 1}(r, \theta, \phi)|^2 dV = \frac{(0.50a_0)^2}{24a_0^5} e^{-0.5a_0/a_0} \left( \frac{3}{8\pi} \sin^2 90^\circ \right) (0.50a_0)^2 (\sin 90^\circ) \times (0.02a_0)(0.00192 \text{ rad})(0.00436 \text{ rad}) = 3.2 \times 10^{-11} \]

(c) Because the probability density associated with any particular state in hydrogen is always independent of \( \phi \), the 2,1,0 probability is again zero and the 2,1,\( \pm 1 \) probability is again \( 3.2 \times 10^{-11} \). (d) The only change in the 2,1,\( \pm 1 \) probability is to replace \( \sin 90^\circ \) with \( \sin 45^\circ \) in three locations, so the new probability is \( (3.2 \times 10^{-11}) (\frac{1}{2^\sqrt{2}})^3 = 1.1 \times 10^{-11} \). For the 2,1,0 probability, the angular factors are the same because \( \cos 45^\circ = \sin 45^\circ \). The only change comes about because of the change from \( 3/8\pi \) to \( 3/4\pi \) in the \( \Theta(\theta) \) term, so the 2,1,0 probability is \( 2.2 \times 10^{-11} \).

12. For \( n = 1, l = 0 \) we have \( P(r) = r^2 |R_{1,0}(r)|^2 = 4r^2 e^{-2r/a_0} / a_0^3 \). To find the maximum, we set \( dP/dr \) to zero:

\[ \frac{dP}{dr} = \frac{4}{a_0^3} \left[ 2re^{-2r/a_0} - r^2 \left( \frac{2}{a_0} \right) e^{-2r/a_0} \right] = \frac{8r}{a_0^3} e^{-2r/a_0} \left( 1 - \frac{r}{a_0} \right) = 0 \]

There are three solutions to this equation: \( r = 0, r = \infty, r = a_0 \). The first two solutions correspond to minima of \( P(r) \); only the solution at \( r = a_0 \) gives a maximum.

13. For \( n = 2, l = 0 \), \( P(r) = r^2 |R_{2,0}(r)|^2 = r^2 \left( \frac{1}{8a_0^3} \left( 2 - \frac{r}{a_0} \right)^2 e^{-r/a_0} \right) = \frac{1}{8a_0^3} \left( 4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2} \right) e^{-r/a_0} \)

Setting \( dP/dr \) to zero, we have

\[ \frac{1}{8a_0^3} r e^{-r/a_0} \left( 8 - \frac{16r}{a_0} + \frac{8r^2}{a_0^2} - \frac{r^3}{a_0^3} \right) = \frac{1}{8a_0^3} r e^{-r/a_0} \left( 2 - \frac{r}{a_0} \right) \left( 4 - \frac{6r}{a_0} + \frac{r^2}{a_0^2} \right) = 0 \]

The five solutions are: \( r = 0, r = \infty, r = 2a_0, r = (3 \pm \sqrt{5})a_0 \). The first three solutions give minima and the last two give maxima.
14. For \( n = 2, l = 1 \), we have \( P(r) = r^2 |R_{21}(r)|^2 = r^2 \frac{1}{24a_0^3} \frac{r^2}{a_0^2} e^{-r/a_0} \). The total probability between \( r = a_0 \) and \( r = 2a_0 \) is

\[
P(a_0 : 2a_0) = \int_{a_0}^{2a_0} P(r) \, dr = \frac{1}{24a_0^5} \int_{a_0}^{2a_0} r^4 e^{-r/a_0} \, dr
\]

We can use Equation 7.4 to evaluate this integral. The result is

\[
P(a_0 : 2a_0) = \frac{1}{24a_0^5} \left[ -a_0^2 e^{-r/a_0} (r^4 + 4a_0 r^3 + 12a_0^2 r^2 + 24a_0^3 r + 24) \right]_{a_0}^{2a_0} = 0.0490
\]

15. \( P(r) \, dr = r^2 |R_{10}(r)|^2 \, dr = (1.00a_0)^2 \frac{4}{a_0^3} e^{-2} (0.01a_0) = 0.0054 \)

16. The kinetic energy is zero where \( E = U \). With the potential energy from Eq. 6.24, we have

\[
E_n = -\frac{me^4}{32\pi^2 e_0^2 \hbar^2 n^2} = -\frac{e^2}{4\pi e_0 r}
\]

\[
r = \frac{8\pi e_0 \hbar^2}{me^2} n^2 = 2a_0 n^2
\]

So the turning points are \( 2a_0 \) for \( n = 1 \), \( 8a_0 \) for \( n = 2 \), and \( 18a_0 \) for \( n = 3 \). The probability densities in Figure 7.10 do change from oscillatory to decreasing exponential at those radial coordinates.

17. The angular probability density is \( P(\theta, \phi) = \frac{15}{8\pi} \sin^2 \theta \cos^2 \theta \). To find the locations of the maxima and minima, we set the derivative equal to zero:

\[
\frac{dP}{d\theta} = \frac{15}{8\pi} (2 \sin \theta \cos^3 \theta - 2 \sin^3 \theta \cos \theta) = \frac{15}{4\pi} (\sin \theta)(\cos \theta)(\cos^2 \theta - \sin^2 \theta) = 0
\]

The three angular terms in parentheses give three sets of solutions: \( \theta = 0, \pi \); \( \theta = \pi/2 \); and \( \theta = \pi/4, 3\pi/4 \). By checking the second derivative, we find that the first two sets give minima (the second derivative is positive) and the third gives maxima (negative second derivative). The angular probability density thus starts at zero along the positive \( z \) direction, rises to a maximum at \( \theta = 45^\circ \), falls again to zero in the \( xy \) plane \( (\theta = 90^\circ) \), rises again to a maximum at \( \theta = 135^\circ \), and finally falls again to zero on the negative \( z \) axis.

18. The angular probability density is \( P(\theta, \phi) = \frac{5}{16\pi} (3 \cos^2 \theta - 1)^2 \). To find the locations of the maxima and minima, we set the derivative equal to zero:
14. For \( n = 2, l = 1 \), we have \( P(r) = r^2 |R_{n,l}(r)|^2 = r^2 \frac{1}{24a_0^3} \frac{r^3}{a_0^3} e^{-r/a_0} \). The total probability between \( r = a_0 \) and \( r = 2a_0 \) is

\[
P(a_0 : 2a_0) = \int_{a_0}^{2a_0} P(r) dr = \frac{1}{24a_0^3} \int_{a_0}^{2a_0} r^4 e^{-r/a_0} dr
\]

We can use Equation 7.4 to evaluate this integral. The result is

\[
P(a_0 : 2a_0) = \frac{1}{24a_0^3} \left[ -a_0 e^{-r/a_0} (r^4 + 4a_0 r^3 + 12a_0^2 r^2 + 24a_0^3 r + 24) \right]_{a_0}^{2a_0} = 0.049024
\]

15. \( P(r) dr = r^2 |R_{1,0}(r)|^2 dr = (1.00a_0)^2 \frac{4}{a_0^3} e^{-2}(0.01a_0) = 0.0054
\]

16. The kinetic energy is zero where \( E = U \). With the potential energy from Eq. 6.24, we have

\[
E_n = -\frac{m e^4}{32\pi^2 \epsilon_0^2 h^2} \frac{1}{n^2} = -\frac{e^2}{4\pi \epsilon_0 r
\]

\[
r = \frac{8\pi \epsilon_0 h^2}{m e^2} n^2 = 2a_0 n^2
\]

So the turning points are \( 2a_0 \) for \( n = 1 \), \( 8a_0 \) for \( n = 2 \), and \( 18a_0 \) for \( n = 3 \). The probability densities in Figure 7.10 do change from oscillatory to decreasing exponential at those radial coordinates.

17. The angular probability density is \( P(\theta, \phi) = \frac{15}{8\pi} \sin^2 \theta \cos^2 \theta \). To find the locations of the maxima and minima, we set the derivative equal to zero:

\[
\frac{dP}{d\theta} = \frac{15}{8\pi} (2 \sin \theta \cos^3 \theta - 2 \sin^3 \theta \cos \theta) = \frac{15}{4\pi} (\sin \theta)(\cos \theta)(\cos^2 \theta - \sin^2 \theta) = 0
\]

The three angular terms in parentheses give three sets of solutions: \( \theta = 0, \pi, \theta = \pi/2 \); and \( \theta = \pi/4, 3\pi/4 \). By checking the second derivative, we find that the first two sets give minima (the second derivative is positive) and the third gives maxima (negative second derivative). The angular probability density thus starts at zero along the positive z direction, rises to a maximum at \( \theta = 45^\circ \), falls again to zero in the xy plane (\( \theta = 90^\circ \)), rises again to a maximum at \( \theta = 135^\circ \), and finally falls again to zero on the negative z axis.

18. The angular probability density is \( P(\theta, \phi) = \frac{5}{16\pi} (3 \cos^2 \theta - 1)^2 \). To find the locations of the maxima and minima, we set the derivative equal to zero:
\[ \frac{dP}{d\theta} = \frac{5}{8\pi} (3\cos^2 \theta - 1)(-6\sin \theta \cos \theta) = -\frac{15}{4\pi} (\sin \theta)(\cos \theta)(3\cos^2 \theta - 1) \]

The three angular terms in parentheses give three sets of solutions: \( \theta = 0, \pi, \frac{\pi}{2} \); and \( \theta = \cos^{-1}(\pm1/\sqrt{3}) = 0.955, 2.186 \). The first two give maxima and the third gives minima. The angular probability density is a maximum on the positive \( z \) axis, falls to zero at \( \theta = 55^\circ \), rises again to a maximum in the \( xy \) plane \( (\theta = 90^\circ) \), falls to zero at \( \theta = 125^\circ \), and rises to a maximum on the negative \( z \) axis.

19. \( P(\theta, \phi) = \frac{3}{16\pi} (3\cos^2 \theta - 1)^2 \)

(b) \( P(\theta, \phi) = \frac{15}{8\pi} \sin^2 \theta \cos^2 \theta \)

(c) \( P(\theta, \phi) = \frac{15}{32\pi} \sin^4 \theta \)

20. (a) degeneracy = \( 2n^2 = 2(5)^2 = 50 \)

(b) For each value of \( l \), the degeneracy is \( 2(2l+1) \).

\[
\begin{align*}
l = 0: & \quad 2(0+1) = 2 \\
l = 1: & \quad 2(2+1) = 6 \\
l = 2: & \quad 2(4+1) = 10 \\
l = 3: & \quad 2(6+1) = 14 \\
l = 4: & \quad 2(8+1) = 18 \\
\text{total:} & \quad 50
\end{align*}
\]

21. \[ \sum_{l=0}^{n-1} 2(2l+1) = 4 \sum_{l=0}^{n-1} l + 2 \sum_{l=0}^{n-1} 1 = 4 \frac{n(n-1)}{2} + 2n = 2n^2 \]
22. (a) $l$ exceeds the maximum permitted value ($n - 1$).

(b) $m_l$ exceeds the maximum permitted value ($l$)

(c) $m_s$ can be only $+1/2$ or $-1/2$

(d) negative values of $l$ are not permitted

23. The selection rule is $\Delta l = \pm 1$, so the $4p$ state can make transitions to any lower $s$ state ($\Delta l = -1$) or $d$ state ($\Delta l = +1$). The possible transitions are then:

$$4p \rightarrow 3s, 4p \rightarrow 2s, 4p \rightarrow 1s, \text{ and } 4p \rightarrow 3d$$

24. (a) The transitions that change $l$ by one unit are

(b) Starting instead with $5d$, the permitted transitions are
25. (a) 7s, 7p, 7d, 7f, 7g, 7h, 7i  
(b) 6p, 6f, 5p, 5f, 4p, 4f, 3p, 2p

26. (a)

(b) Transitions shown with dashed lines violate the $\Delta m_i = \pm 1$ selection rule.

(c) The energy of the initial state is $E_i = E_{3d} + m_i \Delta E$ and the energy of the final state is $E_f = E_{2p} + m_i \Delta E$ (where $\Delta E$ is the spacing between adjacent states). The transition energies can be found from the energy difference:

$$E_i - E_f = (E_{3d} - E_{2p}) + (m_i - m_i) \Delta E = (E_{3d} - E_{2p}) + \Delta m_i \Delta E$$

There are only three permitted values of $\Delta m_i$ (0, ±1), so there are only three possible values of the energy difference: $E_{3d} - E_{2p}, E_{3d} - E_{2p} + \Delta E, E_{3d} - E_{2p} + \Delta E$. 
25. (a) 7s, 7p, 7d, 7f, 7g, 7j  
(b) 6p, 6f, 5p, 5f, 4p, 4f, 3p, 2p

26. (a)

(b) Transitions shown with dashed lines violate the $\Delta m_i = \pm 1$ selection rule.

(c) The energy of the initial state is $E_i = E_{3d} + m_i \Delta E$ and the energy of the final state is $E_f = E_{2p} + m_f \Delta E$ (where $\Delta E$ is the spacing between adjacent states). The transition energies can be found from the energy difference:

$$ E_i - E_f = (E_{3d} - E_{2p}) + (m_i - m_f) \Delta E = (E_{3d} - E_{2p}) + \Delta m_i \Delta E $$

There are only three permitted values of $\Delta m_i$ (0, ±1), so there are only three possible values of the energy difference: $E_{3d} - E_{2p}, E_{3d} - E_{2p} + \Delta E, E_{3d} - E_{2p} + \Delta E$. 
27. (a) In the absence of a magnetic field, the $3d$ to $2p$ energy difference is

$$E = (-13.6057 \text{ eV}) \left(\frac{1}{3^2} - \frac{1}{2^2}\right) = 1.88968 \text{ eV}$$

and the wavelength is

$$\lambda = \frac{hc}{E} = \frac{1239.842 \text{ eV} \cdot \text{nm}}{1.88968 \text{ eV}} = 656.112 \text{ nm}$$

The magnetic field gives a change in wavelength of

$$\Delta \lambda = \frac{\lambda^2 \Delta E}{hc} = \frac{(656.112 \text{ nm})^2}{1239.842 \text{ eV} \cdot \text{nm}} (5.79 \times 10^{-5} \text{ eV/T})(3.50 \text{ T}) = 0.0703 \text{ nm}$$

The wavelengths of the three normal Zeeman components are then 656.112 nm, 656.112 nm + 0.070 nm = 656.182 nm, and 656.112 nm − 0.070 nm = 656.042 nm.

28. The energy of the $2p$ to $1s$ Lyman transition is

$$E = (-13.60570 \text{ eV}) \left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.20428 \text{ eV}$$

and its wavelength (in the absence of fine structure) is

$$\lambda = \frac{hc}{E} = \frac{1239.842 \text{ eV} \cdot \text{nm}}{10.20428 \text{ eV}} = 121.5022 \text{ nm}$$

With the fine structure energy splitting of $4.5 \times 10^{-5} \text{ eV}$, the wavelength splitting is

$$\Delta \lambda = \frac{\lambda^2 \Delta E}{hc} = \frac{(121.5 \text{ nm})^2}{1240 \text{ eV} \cdot \text{nm}} (4.5 \times 10^{-5} \text{ eV}) = 0.00054 \text{ nm}$$

The fine structure splits one level up by $0.5 \Delta \lambda$ and the other down by the same amount, so the wavelengths are

$$\lambda + \frac{1}{2} \Delta \lambda = 121.5024 \text{ nm} \quad \text{and} \quad \lambda - \frac{1}{2} \Delta \lambda = 121.5019 \text{ nm}$$

29. The $3d$ fine structure splitting is roughly

$$\Delta E = mc^2 \alpha^4 \frac{1}{3^3} = 6.0 \times 10^{-6} \text{ eV}$$
27. (a) In the absence of a magnetic field, the $3d$ to $2p$ energy difference is

$$E = (-13.6057 \text{ eV}) \left( \frac{1}{3^2} - \frac{1}{2^2} \right) = 1.88968 \text{ eV}$$

and the wavelength is

$$\lambda = \frac{hc}{E} = \frac{1239.842 \text{ eV} \cdot \text{nm}}{1.88968 \text{ eV}} = 656.112 \text{ nm}$$

The magnetic field gives a change in wavelength of

$$\Delta \lambda = \frac{\Delta \lambda^2}{hc} = \frac{(656.112 \text{ nm})^2}{1239.842 \text{ eV} \cdot \text{nm}} \left( 5.79 \times 10^{-5} \text{ eV/T} \right) (3.50 \text{ T}) = 0.0703 \text{ nm}$$

The wavelengths of the three normal Zeeman components are then 656.112 nm, 656.112 nm + 0.070 nm = 656.182 nm, and 656.112 nm − 0.070 nm = 656.042 nm.

28. The energy of the $2p$ to $1s$ Lyman transition is

$$E = (-13.60570 \text{ eV}) \left( \frac{1}{2^2} - \frac{1}{1^2} \right) = 10.20428 \text{ eV}$$

and its wavelength (in the absence of fine structure) is

$$\lambda = \frac{hc}{E} = \frac{1239.842 \text{ eV} \cdot \text{nm}}{10.20428 \text{ eV}} = 121.5022 \text{ eV}$$

With the fine structure energy splitting of $4.5 \times 10^{-5} \text{ eV}$, the wavelength splitting is

$$\Delta \lambda = \frac{\lambda^2}{hc} \Delta E = \frac{(121.5 \text{ nm})^2}{1240 \text{ eV} \cdot \text{nm}} (4.5 \times 10^{-5} \text{ eV}) = 0.00054 \text{ nm}$$

The fine structure splits one level up by $0.5 \Delta E$ and the other down by the same amount, so the wavelengths are

$$\lambda + \frac{1}{2} \Delta \lambda = 121.5024 \text{ nm} \quad \text{and} \quad \lambda - \frac{1}{2} \Delta \lambda = 121.5019 \text{ nm}$$

29. The $3d$ fine structure splitting is roughly

$$\Delta E = mc^2 \alpha^4 \frac{1}{3^2} = 6.0 \times 10^{-6} \text{ eV}$$
Evaluating the integrals with Equation 7.3, we find

\[ A^2 \left( \frac{2}{(1/a_0)^3} - \frac{1}{a_0 (1/a_0)^2} + \frac{3!}{4a_0^2 (1/a_0)^3} + \frac{1}{4a_0^2 (1/a_0)^3} \right) = 1 \quad \text{or} \quad A = \frac{1}{\sqrt{2a_0^3}} \]

31. For \( n = 2, l = 0 \), we have \( P(r) = r^2 |R_{20}(r)|^2 = r^2 \frac{1}{8a_0^3} \left( 2 - \frac{r}{a_0} \right)^2 e^{-r/a_0} \). The probability to find the electron beyond \( r = 5a_0 \) is

\[
P(5a_0 : \infty) = \int_{r = 5a_0}^{\infty} P(r) dr = \frac{1}{8a_0^3} \int_{r = 5a_0}^{\infty} \left( 4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2} \right) e^{-r/a_0} dr = \frac{1}{8} \int_{r = 5a_0}^{\infty} (4x^2 - 4x^3 + x^4)e^{-x} dx
\]

with \( x = r/a_0 \). The integrals can be evaluated using Equation 7.4. The result of the integration is

\[
P(5a_0 : \infty) = \frac{1}{8} \left( 4 \times 0.2493 - 4 \times 1.5902 + 10.5718 \right) = 0.651
\]

For \( n = 2, l = 1 \), we have \( P(r) = r^2 |R_{21}(r)|^2 = r^2 \frac{1}{24a_0^3} \frac{r^2}{a_0^2} e^{-r/a_0} \). The probability is

\[
P(5a_0 : \infty) = \frac{1}{24a_0^5} \int_{r = 5a_0}^{\infty} r^4 e^{-r/a_0} dr = \frac{1}{24} \int_{r = 5a_0}^{\infty} x^4 e^{-x} dx = 0.440
\]

Thus the \( n = 2, l = 0 \) electron is more likely to be found beyond \( r = a_0 \) than the \( n = 2, l = 1 \) electron.

32. (a)

\[
U = -\frac{e^2}{8 \pi \varepsilon_0 r} = -\frac{e^2}{4 \pi \varepsilon_0} \frac{1}{2r} = -\frac{1.44 \text{ eV} \cdot \text{nm}}{2(0.0529 \text{ nm})} = -27.2 \text{ eV}
\]

\[
K = E - U = -13.6 \text{ eV} - (-27.2 \text{ eV}) = 13.6 \text{ eV}
\]

(b) \( K = 0 \) where \( E = U = -13.6 \text{ eV} \):

\[
U = -13.6 \text{ eV} = -\frac{1.44 \text{ eV} \cdot \text{nm}}{2r} \quad \text{so} \quad r = \frac{-1.44 \text{ eV} \cdot \text{nm}}{2(-13.6 \text{ eV})} = 0.106 \text{ nm} = 2a_0
\]

(c)
\[
P(2a_o; \infty) = \int_{2a_o}^{\infty} r^2 \frac{4}{a_o^2} e^{-2r/a_o} dr = \frac{4}{a_o^2} \left( \frac{r^2}{2/a_o} + \frac{2r}{(2/a_o)^2} + \frac{2}{(2/a_o)^3} \right) \bigg|_{2a_o}^{\infty} = 6.5e^{-4} = 0.238
\]

33. \( r_{av} = \int_0^\infty rP(r) dr = \int_0^\infty r^3 |R_{1,0}(r)|^2 dr = \frac{4}{a_o^2} \int_0^\infty r^3 e^{-2r/a_o} dr = \frac{4}{a_o^2} \frac{3!}{(2/a_o)^4} = \frac{3}{2} a_o
\]

34. 2s level: \( r_{av} = \int_0^\infty rP(r) dr = \int_0^\infty r^3 |R_{2,0}(r)|^2 dr = \frac{1}{8a_o^3} \int_0^\infty r^3 \left( 2 - \frac{r}{a_o} \right)^2 e^{-r/a_o} dr \\
= \frac{1}{8a_o^3} \int_0^\infty \left( 4r^3 - 4\frac{r^4}{a_o^2} + \frac{r^5}{a_o^3} \right) e^{-r/a_o} dr = \frac{1}{8a_o^3} \left( \frac{4}{a_o^4} - \frac{4}{a_o^5} + \frac{5}{a_o^6} \right) = 6a_o
\]

2p level: \( r_{av} = \int_0^\infty rP(r) dr = \int_0^\infty r^3 |R_{2,1}(r)|^2 dr = \frac{1}{24a_o^5} \int_0^\infty r^5 e^{-r/a_o} dr = \frac{1}{24a_o^5} \frac{5!}{(1/a_o)^6} = 5a_o
\]

35. \( U_{av} = \int_0^\infty U(r)P(r) dr = \int_0^\infty \left( -\frac{e^2}{4\pi\varepsilon_0 r} \right) r^2 |R_{1,0}(r)|^2 dr = -\frac{e^2}{4\pi\varepsilon_0 a_o^3} \int_0^\infty e^{-2r/a_o} dr \\
= -\frac{e^2}{4\pi\varepsilon_0 a_o^3} \frac{1}{(2/a_o)^2} = -\frac{e^2}{4\pi\varepsilon_0 a_o^2}
\]

36. Assuming a beam of silver atoms \((m = 108 \text{ u})\) and estimating the magnetic moment of the atom to be about one Bohr magneton, we have

\[
F_z = \mu_z \frac{dB}{dz} = (9.27 \times 10^{-24} \text{ J/T})(10 \text{ T/m}) = 9.3 \times 10^{-23} \text{ N}
\]

The acceleration in the region of the field is

\[
a_z = \frac{F}{m} = \frac{9.3 \times 10^{-23} \text{ N}}{(108 \text{ u})(1.66 \times 10^{-27} \text{ kg/u})} = 518 \text{ m/s}^2
\]

For an oven temperature of 1000 K, the kinetic energy of the atoms is

\[
K = \frac{1}{2} kT = \frac{1}{2} \left( 1.38 \times 10^{-23} \text{ J/K} \right)(1000 \text{ K}) = 2.1 \times 10^{-20} \text{ J}
\]

and the speed of these atoms is
\[
P(2a_0, \infty) = \int_{2a_0}^{\infty} r^2 \frac{4}{a_0} e^{-2r/a_0} dr = \frac{4}{a_0} \left( \frac{2}{2/a_0} \right) \left( \frac{2}{(2/a_0)^2} \right) = 6.5 e^{-4} = 0.238
\]

33. \[
r_{av} = \int_{0}^{\infty} rP(r)dr = \int_{0}^{\infty} r^4 |R_{1,0}(r)|^2 dr = \frac{4}{a_0^3} \int_{0}^{\infty} r^3 e^{-2r/a_0} dr = \frac{4}{a_0^3} \left( \frac{3!}{(2/a_0)^4} \right) = \frac{3}{2} a_0
\]

34. 2s level:
\[
r_{av} = \int_{0}^{\infty} rP(r)dr = \int_{0}^{\infty} r^3 |R_{2,0}(r)|^2 dr = \frac{1}{8a_0^3} \int_{0}^{\infty} r^3 \left( 2 - \frac{r}{a_0} \right)^2 e^{-r/a_0} dr
\]
\[
= \frac{1}{8a_0^3} \int_{0}^{\infty} \left( 4r^3 - 4r^4 + r^5 \right) e^{-r/a_0} dr = \frac{1}{8a_0^3} \left( \frac{4}{3} \frac{3!}{(1/a_0)^4} - \frac{4}{4} \frac{4!}{a_0} (1/a_0)^5 + \frac{1}{5} \frac{5!}{a_0^2} (1/a_0)^6 \right) = 6a_0
\]

2p level:
\[
r_{av} = \int_{0}^{\infty} rP(r)dr = \int_{0}^{\infty} r^3 |R_{2,1}(r)|^2 dr = \frac{1}{24a_0^5} \int_{0}^{\infty} r^5 e^{-r/a_0} dr = \frac{1}{24a_0^5} \frac{5!}{(1/a_0)^6} = 5a_0
\]

35. \[
U_{av} = \int_{0}^{\infty} U(r)P(r)dr = \int_{0}^{\infty} \left( \frac{-e^2}{4 \pi \epsilon_0 r} \right) r^2 |R_{1,0}(r)|^2 dr = \frac{e^2}{4 \pi \epsilon_0 a_0^3} \int_{0}^{\infty} re^{-2r/a_0} dr
\]
\[
= \frac{e^2}{4 \pi \epsilon_0 a_0^3} \frac{4}{(2/a_0)^2} = \frac{e^2}{4 \pi \epsilon_0 a_0}
\]

36. Assuming a beam of silver atoms \((m = 108 \text{ u})\) and estimating the magnetic moment of the atom to be about one Bohr magneton, we have

\[
F_z = \mu_z \frac{dB}{dz} = (9.27 \times 10^{-24} \text{ J/T})(10 \text{ T/m}) = 9.3 \times 10^{-23} \text{ N}
\]

The acceleration in the region of the field is

\[
a_z = \frac{F_z}{m} = \frac{9.3 \times 10^{-23} \text{ N}}{(108 \text{ u})(1.66 \times 10^{-27} \text{ kg/u})} = 518 \text{ m/s}^2
\]

For an oven temperature of 1000 K, the kinetic energy of the atoms is

\[
K = \frac{1}{2} kT = \frac{1}{2} (1.38 \times 10^{-23} \text{ J/K})(1000 \text{ K}) = 2.1 \times 10^{-20} \text{ J}
\]

and the speed of these atoms is
\[
v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(2.1 \times 10^{-20} \text{ J})}{(108 \text{ u})(1.66 \times 10^{-27} \text{ kg/u})}} = 480 \text{ kg/s}
\]

The time for an atom to travel 1 meter through the magnetic field region is

\[
t = \frac{1 \text{ m}}{480 \text{ m/s}} = 2.1 \times 10^{-3} \text{ s}
\]

Let the atoms enter the field with \( z = 0 \) and \( v_z = 0 \). Then after passing through the 1-meter field region,

\[
z = \frac{1}{2} a_z t^2 = \frac{1}{2} (518 \text{ m/s}^2)(2.1 \times 10^{-3} \text{ s})^2 = 1.1 \text{ mm}
\]

\[
v_z = a_z t = (518 \text{ m/s}^2)(2.1 \times 10^{-3} \text{ s}) = 1.1 \text{ m/s}
\]

After leaving the region of the field there is no longer an acceleration, but the \( z \) component of the velocity causes an additional displacement in the \( z \) direction. The horizontal velocity is unchanged, so the atom takes \( 2.1 \times 10^{-3} \text{ s} \) to pass through the field-free region, and the additional displacement is

\[
z = v_z t = (1.1 \text{ m/s})(2.1 \times 10^{-3} \text{ s}) = 2.2 \text{ mm}
\]

The total displacement is then \( 1.1 \text{ mm} + 2.2 \text{ mm} = 3.3 \text{ mm} \). We would thus expect to see images on the screen separated by a few mm.

37. For 1s: \((r^{-1})_{av} = \int_0^\infty r^{-1} P(r) \, dr = \int_0^\infty r^2 |R_{1,0}(r)|^2 \, dr = \frac{4}{a_0^2} \int_0^\infty r e^{-2r/a_0} \, dr = \frac{4}{2a_0^2} \frac{1}{a_0^2} = \frac{1}{a_0^2}
\]

For 2s:

\[
(r^{-1})_{av} = \int_0^\infty r^2 |R_{2,0}(r)|^2 \, dr = \frac{1}{8a_0^3} \int_0^\infty r^2 \left(2 - \frac{r}{a_0}\right)^2 e^{-r/a_0} \, dr
\]

\[
= \frac{1}{8a_0^3} \left\{ \frac{4r^3}{a_0} + \frac{r^3}{a_0^2} \right\} e^{-r/a_0} = \frac{1}{8a_0^3} \left(4 \frac{r}{a_0} \frac{1}{(1/a_0)^2} - 4 \frac{2}{a_0} \frac{1}{(1/a_0)^3} + \frac{3!}{a_0} \frac{1}{(1/a_0)^4} \right) = \frac{1}{4a_0^4}
\]

For 2p:

\[
(r^{-1})_{av} = \int_0^\infty r^2 |R_{2,1}(r)|^2 \, dr = \frac{1}{24a_0^4} \int_0^\infty r^2 e^{-r/a_0} \, dr = \frac{1}{24a_0^4} \frac{3!}{(1/a_0)^4} = \frac{1}{4a_0^4}
\]

38. (a) In Cartesian coordinates, the solution can be found using Eq. 7.9, which is separable for this potential energy. The solutions can be written

\[
\psi_{n,n,n}(x,y,z) = X_{n_x}(x)Y_{n_y}(y)Z_{n_z}(z) , \text{ and for each of the 3 functions there is an energy of } (n_i + \frac{1}{2})\hbar\omega , \text{ where } i = x, y, \text{ or } z . \text{ The total energy is then } E_{n,n,n} = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega .
\]
(b) Ground state: \( E = \frac{1}{2} \hbar \omega \), \((n_x n_y n_z) = (000)\)

1\(^{st}\) excited state: \( E = \frac{1}{2} \hbar \omega \), \((n_x n_y n_z) = (100), (010), (001)\)

2\(^{nd}\) excited state: \( E = \frac{1}{2} \hbar \omega \), \((n_x n_y n_z) = (200), (020), (002), (110), (101), (011)\)

(c) The ground state has \( n = 0 \) and therefore only \( l = 0 \), so it is non-degenerate. The first excited state has \( n = 1 \) and therefore \( l = 1 \), which has degeneracy 3 \((m_l = +1, 0, -1)\). The second excited state has \( n = 2 \) and therefore includes \( l = 0 \) (degeneracy = 1) and \( l = 2 \) (degeneracy = 5), for a total degeneracy of 6.

39. (a) The radial dependence of the 3 functions is \( e^{-r/a_0} \) for \( n = 1, l = 0 \); \( re^{-r/2a_0} \) for \( n = 2, l = 1 \); and \( r^2 e^{-r/3a_0} \) for \( n = 3, l = 2 \). A guess for the next function in the series is \( r^3 e^{-r/4a_0} \) for \( n = 4, l = 3 \).

(b) The necessary derivatives are:

\[
\begin{align*}
\frac{dR}{dr} &= 3r^2 e^{-r/4a_0} - r^3 e^{-r/4a_0} / 4a_0 \\
\frac{d^2 R}{dr^2} &= 6re^{-r/4a_0} - 3r^2 e^{-r/4a_0} / 4a_0 - 3r^2 e^{-r/4a_0} / 4a_0 + 3e^{-r/4a_0} / 16a_0^2
\end{align*}
\]

Substituting into Eq. 7.12 and canceling the common exponential factor gives:

\[
- \frac{\hbar^2}{2m} \left[ 6r - \frac{3r^2}{2a_0} + \frac{r^3}{16a_0^2} + \frac{2}{r} \left( 3r^2 - \frac{r^3}{4a_0} \right) \right] + \left( \frac{-e^2}{4\pi\varepsilon_0} + \frac{12\hbar^2}{2mr^2} \right) r^3 = Er^3
\]

The terms linear in \( r \) cancel, leaving us with

\[
\frac{\hbar^2 r^2}{2ma_0} = \frac{\hbar^2 r^3}{32ma_0^2} = \frac{e^2 r^2}{4\pi\varepsilon_0} = Er^3
\]

When the value of \( a_0 \) is inserted, the 2 terms in \( r^2 \) cancel, leaving us with

\[
E = \frac{\hbar^2}{2m} \frac{1}{16a_0^2} = \frac{\hbar^2}{16} \left( \frac{me^2}{4\pi\varepsilon_0\hbar^2} \right)^2 = -\frac{1}{16} \frac{me^4}{32\pi^2\varepsilon_0^2\hbar^4}
\]

which is the expected energy for the \( n = 4 \) state.

40. (a)

\[
P(0: R) = \int_0^R r^2 \left( \frac{4}{a_0^3} e^{-2r/a_0} \right) dr = \left. -\frac{4}{a_0^3} e^{-2r/a_0} r^2 + \frac{2}{2/a_0} + \frac{2}{4/a_0^2} \right|_0^R
\]

\[
= 1 - e^{-2R/a_0} \left( \frac{2R^2}{a_0^2} + \frac{2R}{a_0} + 1 \right)
\]